Phase ordering dynamics in a ferromagnetic spin-1 Bose-Einstein condensate

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Abstract

Spinor Bose-Einstein condensates exhibit both superfluid and magnetic order, and accommodate phases with rich symmetry properties and topological defects. Transitions between these phases can be induced by tuning external fields. In this thesis we explore the dynamics of order formation in a quasi-2D spin-1 ferromagnetic condensate following a quench from an unmagnetised phase to one of three ferromagnetic phases. The ferromagnetic phases exhibit distinct symmetry properties (easy-plane, easy-axis or isotropic) and support distinct topological defects. In each phase we observe scale invariant ordering and identify the relevant topological defect affecting the order parameter growth. We find that each phase is characterised by a distinct dynamic critical exponent. In the easy-plane phase we identify a persistent turbulent cascade that affects spin ordering long after all topological defects have annihilated. In addition to our exploration of phase ordering dynamics, we study a microscopic model of spin vortex dynamics in the easy-plane phase. Our work provides a comprehensive theoretical study of phase ordering in a conservative system, provides a thorough foundation for studies of phase ordering in antiferromagnetic and higher spin condensates, and offers prospects for further research into fundamental questions regarding the ordering properties of spin systems. Our work is pertinent to current experiments, which have explored the initial stages of phase ordering in both ferromagnetic and antiferromagnetic spin-1 condensates.
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Chapter 1

Introduction

1.1 Continuous phase transitions

The concept of a phase transition is ubiquitous in physics. Transitions between solids, liquids and gases are the most familiar examples, and are utilised throughout the natural and technological world. More exotic phase transitions exist in a wide variety of physical systems: ionization in stars; magnetization in iron; spinodal decomposition in material construction; emergence of superconductivity and Bose-Einstein condensation at low temperatures; breaking of symmetries in the early universe; changing of the crystaline structures in a solid; and many, many more. The variety and ubiquity of phase transitions provides a rich field for exploration, as researchers in diverse fields compare and contrast phase transitions between vastly different systems. Phase transitions are a many body phenomena (usually) requiring interactions and involving a wide range of length scales. Understanding such physics at the microscopic level is exceedingly challenging, and so a central goal in the study of phase transitions is identifying emergent simpler physics that can describe observations at the macrscopic level.

Phase transitions fall into two groups: first order and continuous. A first order phase transition involves a latent heat. Latent heat is connected with a major reordering of the particles in the system, for example the crystallisation of a solid from a liquid. A quantitative theory that describes accurately first order phase transitions at the macroscopic level has not yet been developed, and such phase transitions will not be the focus of this thesis. A continuous phase transition requires no absorption or emission of latent heat and therefore the ordering of particles changes continuously. Unlike first order phase transitions, a quantitative macroscopic theory describing continuous phase transitions does exist. This is based on the discovery that
systems with a continuous phase transition exhibit \textit{scale invariance} close to the transition, which gives rise to a diverging correlation length and diverging susceptibilities. The scale invariant properties of the transition are independent of the microscopic details of the system and instead depend on general properties of the system such as symmetries, dimension and conservation laws. This major simplification allows for quantitative studies of continuous phase transitions. Such studies involve identifying the important scale invariant behaviour of a transition and grouping microscopically distinct systems into \textit{universality classes} that share the same scale invariant behaviour. A continuous phase transition in a phase diagram is also called a \textit{critical point}, and the collection of behaviour arising from the scale invariance of a system close to a critical point is termed \textit{critical phenomena}. The celebrated theory that describes critical phenomena is known as the \textit{renormalization group}, which will be discussed in Sec. 2.7.

The history of critical points is a rich one, beginning in the mid-late 19th century with studies of the liquid-gas transition and ferromagnetism [1–9], and culminating with the sophisticated ideas of the renormalization group a century later [10–13].\(^1\) This first century of research focused mainly on the equilibrium properties of critical points. The field was extended to include dynamics around critical points in seminal works by Hohenberg and Halperin [15–19]. In this extension both spatial correlation lengths and temporal correlation times diverge close to a critical point. This early work employed hydrodynamic arguments, based on assumptions of local equilibrium. The low frequency dynamics in the system can then be related to the equilibrium properties of long wavelength degrees of freedom. One outcome of the work of Hohenberg and Halperin is the Kibble-Zurek mechanism [20–26], which describes the behaviour of topological defects crossing a critical point and was one of the early examples connecting early universe and condensed matter physics. In later work, it was found that ideas of dynamic critical phenomena were more general, and could be applied to the relaxation dynamics of systems far from equilibrium [27].\(^2\) This led to the theory of phase ordering dynamics [37, 38], which describes the ordering of systems

\(^1\)For a detailed discussion of this history, see [14].

\(^2\)Even now the precise regimes where dynamic critical phenomena can be applied is subject to debate. The notion of \textit{nonthermal fixed points} has been introduced [28–32] to alleviate some of this debate. A nonthermal fixed point is analogous to a critical point, but appears in the dynamic field theory of a system. Ideas of universality and diverging correlation lengths and time still apply to such a point, but it is not necessary to assume the system is close to equilibrium [33–35]. We will discuss this in more detail in Sec. 2.7. It is worth noting also that very recent work has provided evidence for first order phase transitions exhibiting dynamic critical behaviour [36].
quenched across a critical point. It is this theory of phase ordering dynamics that will be the focus of this thesis.

1.2 Phase ordering dynamics

In this section we briefly review the theory of phase ordering dynamics. We will discuss this theory in much more detail in Chap. 2. The theory of phase ordering dynamics, as reviewed by Bray [37, 38], considers how systems quenched across a critical point to an ordered phase evolve toward equilibrium. The basic ingredients of the theory are the following:

1. In the early dynamics, the quenched system locally orders, with the length scale of ordering determined by microscopic details. This involves a local breaking of symmetry in the system. Causally disconnected regions of the system will break symmetry differently, Fig. 1.1(i), and this leads to the formation of domains.

2. The domains proceed to grow and compete for the global equilibrium state. Associated with this growth is the annihilation of so-called topological defects. We will define what we mean by topological defects in Sec. 2.6.

3. Once the domains become larger than microscopic length scales, their growth becomes scale invariant, Fig. 1.1. At any given time $t$, rescaling the domain sizes by a suitably chosen growing length scale $L(t)$ leads to a domain size distribution that is independent of time. To put simply: evolving the system forward in time has the same affect as if we stretched the system in space. Furthermore, the growing length scale $L(t)$ grows as a power law with time, $L \sim t^{1/z}$ where $z$ is the dynamic critical exponent of the system.

4. Systems exhibiting the same dynamic critical behaviour share the same dynamic critical exponent and belong to the same dynamic universality class. The dynamic critical behaviour is determined by general properties of the system, such as symmetries, dimension and conservation laws, rather than microscopic details.

The theory of phase ordering dynamics was expounded by Bray and collaborators, amongst others, in studying the formation of order in spin models [39–54]. The theory of phase ordering dynamics has been applied to a number of different systems, for example: spinodal decomposition in binary fluids [55–62]; interface growth [63–66]; nematic liquid crystals [67–
Figure 1.1: Simulation results for phase ordering dynamics in the Ising model following a temperature quench from above to below the Curie temperature, in zero external magnetic field. Images of the system at successive times are shown by frames (i)-(iv). Black regions indicate spins of one direction and white regions indicate spins of opposite direction. (i) Noise in the system seeds the growth of small domains with different order parameter directions. (ii)-(iv) The domains grow as the system evolves toward an ordered state. The domain distribution at later times resembles the distribution at earlier times, but with a rescaling of lengths. The growth of domains is therefore scale invariant. Figure adapted from [37].

76]; cosmology [77–82]; fractal clusters [83]; and even neuron function in neurology [84] and opinion spreading in sociology [85–94]. Like with many condensed matter systems, much of these previous studies of phase ordering dynamics have invoked phenomenological models of systems rather than accurate microscopic models. Although such studies are fruitful, especially considering the robustness of critical phenomena against microscopic details, it is of interest to explore the theory in systems where microscopic studies are tractable.

1.3 Bose-Einstein condensates

A many particle system that has seen an explosion of research in the last 20 years is gaseous Bose-Einstein condensates. A Bose-Einstein condensate (BEC), originally proposed by Bose and Einstein in 1924 [95–98], is a phase of matter with macroscopic occupation of a single particle state. The phase arises from the indistinguishability of identical particles. Since the particles are required to occupy the same state, the phase is only accessible for particles with Bose statistics. The presence of a BEC was suggested to play a role in superfluidity in liquid helium [99] and superconductivity in cooled metals [100, 101], however the effect is disguised by strong interactions. In 1995, groups at JILA [102], MIT [103] and Rice [104–106] produced the first gaseous BECs
in the laboratory, using rubidium, sodium and lithium respectively. This monumental achievement inspired a wave of both experimental and theoretical research into the behaviour of BECs that has continued until this day. The low density\(^3\) of gaseous BECs allows for almost all atoms to condense into the BEC phase. Combined with trapping techniques that isolate the system from the environment, this provides an exceptionally clean many body system. In addition, a low density BEC can be described by accurate and tractable theoretical models. Dilute gaseous BECs are therefore ideal systems for both experimental and theoretical studies of BEC properties and more general many body physics. Such research includes studies of: matter wave interference [108–113]; topological defects such as quantized vortices [114–123], solitons [124–139] and more exotic defects [140–153]; Anderson localisation [154–159]; supersolids [160–166]; quantum turbulence [167–177]; and microscopic studies of thermalization [178–186]. Many of these explorations would be inaccessible in more complex many body systems.

Of most relevance to this thesis are the many studies on BEC dynamics during or after a quench across a critical point. This has included studies of the condensation transition itself [187–196], but also transitions within a BEC, such as Mott-insulator transitions [197–205], miscibility transitions in binary condensates [206–219], percolation [220–222], transitions in spinor condensates [223–235], and others [236–238]. Studying the dynamics of continuous phase transitions within a gaseous BEC inherits the theoretical and experimental advantages of this system over other many body systems. In addition, transitions within a BEC are of special interest, as they fall into the class of quantum phase transitions, which are phase transitions that exist even at zero temperature. The diverging properties of a continuous quantum phase transition arise from diverging quantum fluctuations, rather than diverging thermal fluctuations. Exploring connections between quantum and thermal continuous phase transitions has proved useful, with much of the theory of thermal transitions carrying over into the quantum regime [239].

1.4 Spinor Bose-Einstein condensates

Bose-Einstein condensates contain bosons, which necessarily have integer spin. The first gaseous BEC experiments trapped the condensed atoms in a single hyperfine spin level using magnetic fields. The strong Zeeman shifts caused by the trapping field would freeze out any spin changing collisions

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\(^3\)The important quantity is the gas parameter \(na^3\) where \(n\) is the density and \(a\) is the atomic scattering length. If this quantity is much smaller than unity, the system can be regarded as dilute [107].
so that the atoms in the condensate were described by a scalar field. Soon after these early experiments, a BEC was formed using an optical trap, which allowed the condensed atoms to occupy different spin states \[240–242\]. The condensate is then described by a spinor field. These early spinor condensates utilised the spin-1 hyperfine levels of alkali atoms\(^4\) \(^{23}\)Na and \(^{87}\)Rb. More recently, higher spin condensates have also been produced \[243–250\]. The accessibility of different spin states gives rise to rich phase diagrams, as the condensed atoms form different spin ordering depending on weak control fields and interparticle interactions \[251–260\]. Spin systems exhibit either ferromagnetic or antiferromagnetic order, depending on the sign of the spin interactions. This has led to a range of research relevant to phase transition dynamics in ferromagnetic and antiferromagnetic spin-1 and higher spin condensates. For example, experimental and theoretical work has explored: topological defects \[261–273\]; spin turbulence \[169, 274–281\]; Berezinskii-Kosterlitz-Thouless transitions; \[267, 282\]; the Kibble-Zurek mechanism \[224, 225, 227, 229, 230, 232, 234\]; instabilities and symmetry breaking following a quench \[223–227, 283–294\]; and phase ordering dynamics \[228, 231, 295–297\]. Of particular relevance to this thesis are the results on symmetry breaking and phase ordering dynamics in ferromagnetic spin-1 condensates, which we will now briefly discuss.

### 1.5 Phase ordering dynamics in \(^{87}\)Rb

A spin-1 ferromagnetic condensate can be tuned between four different phases by changing an external quadratic Zeeman field, see Fig. 1.2(a). The four phases (polar, easy-plane, isotropic and easy-axis) possess different magnetic properties, which we will discuss in more detail in Sec. 3.7. Symmetry breaking and dynamic instabilities were observed in the easy-plane phase in beautiful experiments by Stamper-Kurn’s UC Berkeley group \[223, 283\]. In this work, an \(^{87}\)Rb condensate was quenched from the unmagnetized (polar) phase to the easy-plane phase. The choice of spin direction leads to the formation of symmetry breaking magnetic domains. The domains can then be imaged \textit{in situ} using polarization dependent phase contrast imaging \[298\]. Figure 1.2(b) shows \textit{in situ} images of the transverse magnetic domains following the quench. In later work \[295\] the Berkeley group explored quenches to the easy-axis and

\(^4\)The number of spin states available to a multi-electron atom is very large, consisting of all the combined spins of subatomic particles. The spin-1 manifold is achieved in \(^{23}\)Na and \(^{87}\)Rb by combining the nucleus in the spin-3/2 state and the outer electron in the spin-1/2 state in the combination 3/2-1/2=1, with all inner electrons paired in the spin-0 state. Note that a spin-2 condensate can also be obtained from the combination 3/2+1/2=2.
isotropic phases, identifying symmetry breaking domains across the ground state manifold for each quench, see Fig. 1.3(a). Signs of initial domain coarsening was also observed, see Fig. 1.3(b).

The Berkeley experiments inspired a wave of theoretical work to explain in detail the symmetry breaking dynamics observed [224–227, 284, 285]. This work was largerly concerned with the early time dynamics of unstable modes following a quench to one of the three magnetized phases. Long time phase ordering dynamics in the easy-axis phase was explored by Mukerjee et al. [228], but with somewhat inconclusive results. A more careful analysis was done by Kudo and Kawaguchi [231], who identified that the long time phase ordering in the easy-axis phase behaves like a classical immiscible binary fluid (see also [217]). Kudo and Kawaguchi also explored a specific case of long time Mermin-Ho vortex dynamics in the easy-plane phase [297]. Simulations by Barnett et al. of easy-plane phase ordering suggested there was an extended period of coarsening dynamics following the early time symmetry breaking dynamics [296], but the nature of the coarsening dynamics was not explored. This limited research has left many questions about the long time phase ordering of spin-1 ferromagnetic condensates, and the problem has been identified as a significant one in the field [299]. In this thesis we provide a comprehensive study of the long time phase ordering dynamics following quenches in a spin-1 ferromagnetic condensate.

1.6 Outline

The outline of the thesis is as follows. Chap. 2 presents relevant background material on phase transitions, critical points and phase ordering dynamics. Chap. 3 presents background material on spin-1 condensates and early time dynamics following quenches in spin-1 ferromagnetic condensates. Chap. 4 presents results for the long time phase ordering dynamics following quenches to the easy-plane and easy-axis ferromagnetic phases. We identify the role that topological defects play in the phase ordering dynamics and identify the dynamic universality class of each phase. Chap. 5 presents results for the long time phase ordering dynamics following a quench to the isotropic ferromagnetic phase. In this phase we identify \( \mathbb{Z}_2 \) vortices, which annihilate in a way consistent with the phase ordering. For a weakly anisotropic system we observe a dynamic transition from isotropic to anisotropic coarsening, which occurs at a critical domain size. In Chap. 6 we study late stages of thermalisation in the easy-plane phase. We find that the late stages of thermalisation is not controlled by defect annihilation. Instead, thermalisation is controlled by an extremely slow spin wave cascade that moves to longer
Figure 1.2: (a) Ground state phase diagram of a spin-1 ferromagnetic condensate for varying quadratic Zeeman energy $q$. For large quadratic Zeeman energy there is no magnetization, which is termed the polar phase. A critical point occurs at $q = q_0$, where $q_0$ is on the order of the spin interaction energy. For $0 < q < q_0$ the ground state magnetization points somewhere in a plane transverse to the external field. This phase is termed the easy-plane phase. For $q = 0$ the magnetization points anywhere on the surface of a sphere. This phase is termed the isotropic phase. For $q < 0$ the magnetization points in either the same or opposite direction to the external field. This phase is termed the easy-axis phase. (b) In situ images of easy-plane magnetization for successive times following a quench from the polar phase to the easy-plane phase, from [223]. Times (in ms) are indicated above the panels. The colours show domains with different spin direction, according to the colour wheel in the bottom left corner of the image. Darker regions correspond to lower spin density. This symmetry breaking occurs during the first part of phase ordering dynamics.
Figure 1.3: Experimental observation of domain formation following a quench in temperature to the easy-axis ($q < 0$), isotropic ($q = 0$) and easy-plane ($q > 0$) phases. (a) Transverse (top) and longitudinal (bottom) spin domains for evolution times ranging from 0.25 s to 2.0 s, for quenches to each of the three phases. Colour maps are shown at the right of the figure. The Zeeman field is chosen to point along the $F_x$ direction in this experiment. Darker regions correspond to lower spin density. (b) Growth of average domain size versus time for dynamics in the isotropic phase. Figure adapted from [295].
length scales in a scale invariant way. Coupling the system to a reservoir destroys the slow spin wave cascade, highlighting a substantial difference between the microcanonical and grand canonical dynamics. In Chap. 7 we simulate the microscopic dynamics of polar-core spin vortices, which are the topological defects in the easy-plane phase, and compare this with an analytic model derived using variational Lagrangian methods. We conclude in Chap. 8, identifying a number of further areas of research that are motivated by the work presented here.

1.7 Papers arising from this work

The following papers have arisen from this work.


- Lewis A. Williamson and P. B. Blakie. Thermalisation in an easy-plane ferromagnetic superfluid. *In preparation*. (Chap. 6.)
Chapter 2

Background: phase transitions and phase ordering dynamics

2.1 Statistical mechanics

The macrostate of a thermodynamic system can be specified by extensive thermodynamic variables \( V_i \), such as energy, particle number, volume, magnetization etc. By “extensive” we mean variables that increase in proportion to the system size. Although it is possible to carry out calculations in terms of these extensive variables, it can prove easier to instead introduce new variables that reflect how the system reacts to changes in the extensive variables. For each variable \( V_i \) we introduce a quantity \( \beta_i \) that is related to the entropy \( S \) of the system by,

\[
\beta_i = \left( \frac{\partial S}{\partial V_i} \right)_{V_{\neq i}, \text{fixed}}. \tag{2.1}
\]

We term the quantities \( \beta_i \) “generalised forces”, since an imbalance in a particular \( \beta_i \) between two systems drives a displacement of the corresponding variable \( V_i \). The particular \( \beta_i \) corresponding to energy is the inverse temperature. The others behave as “generalized temperatures”, however we will not use this term to avoid confusion with the real temperature of the system. The generalised forces are ratios of two extensive quantities and so are intensive.\(^1\)

\(^1\)We have introduced all the generalised forces on an equal footing. For the present discussion this is permissible, however in general it can be misleading. In particular, the particular \( \beta_i \) corresponding to a system’s temperature is special. To see this, we write the total energy of a system as \( E = \sum p_k E_k \) where \( E_k \) is the energy of state \( k \) and \( p_k \) is the probability the system will be found in this state. Changes in energy then take the form \( dE = \sum p_k dE_k + \sum dp_k E_k \). The first term corresponds to work done on the system,
The entropy is a function of the extensive variables $V_i$ and is maximised in equilibrium. In many cases the generalised forces $\beta_1, \ldots, \beta_n$ are fixed, e.g. if the system is coupled to a reservoir, in which case it can be more convenient to consider a function of the generalised forces $\beta_i$ known as the free energy. Precisely, we define the free energy as the Legendre transform of the entropy [305],

$$F(\beta_1, \ldots, \beta_n) = -S(V_1, \ldots, V_n) + \sum_i \beta_i V_i,$$

where the $V_1, \ldots, V_n$ are understood as functions of the generalised forces. The usefulness of this definition can be seen by considering a system coupled to a reservoir with $\beta_i$ equal to that of the system, allowing exchange of $V_i$ between the system and reservoir. In equilibrium (maximum entropy of both the system and reservoir) the free energy assumes its minimal value, where the minimisation is over the extensive variables $V_i$ with fixed $\beta_i$ [306]. Following this minimum as a function of the $\beta_i$ then maps out $F(\beta_1, \ldots, \beta_n)$, from which various quantities of interest can be computed (see below).

A related quantity of interest is the partition function, defined as

$$Z \equiv \sum_s \exp \left( -\sum_i \beta_i V_i(s) \right),$$

where the sum is over all possible system states $s$. The free energy (2.2) is related to the partition function by [304],

$$F = -\ln Z,$$

where the quantities $V_i$ and $S$ in Eq. (2.2) are understood as averages. The partition function arises naturally for a system coupled to a reservoir with fixed $\beta_i$. In such a case, the probability of the system being found in state $s$, which requires borrowing quantities $V_1, \ldots, V_n$ from the reservoir is,

$$P_s = \frac{\exp (-\sum_i \beta_i V_i(s))}{Z},$$

Knowing the partition function (or equivalently the free energy) allows one to compute many quantities of interest. For example, the expectation value through quasistatic changes of particle number, volume, etc. The second term corresponds to heat exchange [304]. For quasistatic processes, doing work on a system does not change the system’s entropy. However, heat exchanges do. Therefore heat and correspondingly temperature play a unique role.

\[\text{This is really the free energy divided by the temperature. This definition proves useful for the present discussion. We also set Boltzmann’s constant } k_B \text{ to unity for now.}\]
of a thermodynamic variable \( V_i \) of a system coupled to a reservoir is,

\[
\langle V_i \rangle = -\frac{1}{Z} \frac{\partial Z}{\partial \beta_i} = \frac{\partial F}{\partial \beta_i},
\]

(2.6)

while fluctuations away from the mean are,

\[
\langle V_i^2 \rangle - \langle V_i \rangle^2 = \frac{\partial^2 F}{\partial \beta_i^2}.
\]

(2.7)

It is possible to show that the description above is also applicable to a sufficiently large closed system, i.e. with no reference to a reservoir. Roughly speaking, we apply the above treatment to a small region of a large system with the remainder of the system treated as a reservoir. This small region may exhibit fluctuations in observable quantities, however relative fluctuations across the total system will be suppressed. This treatment can be made mathematically precise, but is a little involved and so we give it in Appendix A.

### 2.2 Thermal phase transitions

Broadly speaking, a thermal phase transition arises when small changes of the generalized forces \( \beta_i \) of a system lead to large changes in the properties of the system. For example, when liquid water at atmospheric pressure is cooled slightly below 0°C, the water molecules develop into a rigid crystalline structure, see Fig. 2.1. As a result, properties such as density, heat conduction and refractive index change abruptly. We specify phase transitions as “thermal” here to distinguish from quantum phase transitions, which we will discuss in Sec. 2.5.

Mathematically, a thermal phase transition occurs at nonanalytic points in the partition function \( Z \). Note that for nonanalytic points to arise, we must necessarily be working in the thermodynamic limit, i.e. system size \( \to \infty \) while keeping the density of particles fixed.\(^3\) For interacting many-body systems, the exact partition function is in general very difficult to compute.\(^4\) This thesis will be concerned with a certain type of phase transition known as a continuous phase transition. Fortunately, continuous phase transitions exhibit many properties that are independent of precise microscopic details, and instead depend on more general properties of the system such as symmetries,
Figure 2.1: The solid and liquid phases of water. The two phases exhibit differences in rigidity, density, heat conduction, refractive index, etc.. In the photo, the solid water is rigid and so is unaffected by a gentle wind, whereas the nonrigid liquid water is contorted by the wind. A phase transition between solid and liquid water exists at 0°C at atmospheric pressure. (Image from shutterstockfootage.com.)
dimension and conservation laws. Therefore it is feasible that a simplified partition function reflecting only these general properties may capture the important scale invariant features of continuous phase transitions.

Such a formulation was presented by Landau in 1937 [309]. Landau had recognised that phase transitions involve a change in the symmetry of a system. For example, liquid water possesses a continuous translational symmetry, whereas ice possesses only a discrete symmetry dependent on the lattice structure. Another example is the development of a magnetisation direction in iron when it is cooled below its Curie temperature, which changes the rotational symmetry of the system. As Landau identifies, “Elements of a symmetry are either present or absent; no intermediate case is possible” [310]. This abrupt change in symmetry is associated with a phase transition. The process of reducing the symmetries of a system due to a phase transition, termed symmetry breaking, corresponds to an ordering of the system. To describe phase transitions generally, Landau introduced the notion of an order parameter $\phi$, which is some quantity (a scalar, vector or more general tensor) that is zero on the disordered side of a phase transition and nonzero on the ordered side. For example, in the case of iron we would choose the order parameter to be the magnetisation (a vector), which would have zero amplitude above the Curie temperature and nonzero amplitude below the Curie temperature. Landau formulated a theory of phase transitions based on free energies that are simple functions of $\phi$. This provided a starting point for a quantitative theory of continuous phase transitions, as well as a qualitative understanding of first order phase transitions.

2.3 Landau’s theory of phase transitions

A full treatment of phase transitions using the partition function as introduced in Sec. 2.1 is in general difficult. As discussed in the previous section, Landau had observed that phase transitions involve changes of symmetry that can be captured by some order parameter $\phi$. Landau utilised this insight to formulate a treatment of the partition function that is more amenable to the study of phase transitions: rather than considering all possible microstates of a system, one instead considers only those that modify the order parameter $\phi$. The statistics of such microstates will represent the statistics of the

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5Landau’s theory of symmetry changes does not encompass all phase transitions. Some phase transitions are instead best understood through changes in a system’s topology (we will mention these again in Sec. 2.6.3). For the work presented in this thesis, Landau’s ideas of symmetry breaking give a sufficient intuition of the physics behind phase transitions. In Chap. 8 we will mention possible further research relating to topological phase transitions.
symmetry of the system and therefore the statistics of the phase transition. We represent each microstate by an order parameter field $\phi(x)$. The spatial dependence of $\phi$ allows for spatial fluctuations of order, which will be present at finite temperature. (In Landau’s original treatment the spatial variations of the order parameter were ignored. We will discuss this case shortly.) The probability of the system being found in a state $\phi(x)$ is,

$$P_\phi = \frac{\exp(-H(\phi))}{Z},$$

(2.8)

where

$$H \equiv \sum_i \beta_i V_i(\phi)$$

(2.9)

and

$$Z \equiv \sum_\phi \exp(-H(\phi)).$$

(2.10)

Equations (2.8) and (2.10) take the same form as Eqs. (2.5) and (2.3) respectively. Progress can be made by choosing a simple phenomenological functional $H(\phi)$ based on symmetries of the system. Such an approach can be effective at describing continuous phase transitions at a quantitative level, since properties of such transitions depend on general features of the system rather than precise microscopic details.

A qualitative understanding of both continuous and first order phase transitions can be obtained by making a bold approximation: we retain in the sum (2.10) only a single uniform field $\bar{\phi}$. For any given values of external parameters $\beta_i$, we can construct a free energy,

$$F(\bar{\phi}) = -\ln Z = H(\bar{\phi}).$$

(2.11)

A schematic of a possible free energy $F(\bar{\phi})$ is plotted in Fig. 2.2(a). Here we plot $F$ against a scalar $\bar{\phi}$ but this could be generalised using higher dimensional plots. The optimal choice of the field $\bar{\phi}$ is the one which minimises the free energy $F(\bar{\phi})$. Changing the $\beta_i$ will generally change the shape of this free energy curve. When the nature of the free energy minimum is changed in a nonanalytic way, the system exhibits a phase transition. Fig. 2.2(b)-(d) show examples of free energy curves that are useful for understanding general properties of first order and continuous phase transitions. We describe these three cases below.
Figure 2.2: Representative free energy profiles $F$ as a function of a uniform order parameter $\phi$. (a) An arbitrary free energy profile. The equilibrium state corresponds to the $\phi$ that minimises $F$. (b) A first order phase transition. Changing a generalized force $\beta_1$ changes the free energy curve from blue to red. As a result, there is an abrupt change in the minimum of $F$, resulting in a first order phase transition. (c) The free energy curve at a first order phase boundary. (d) A continuous phase transition. From the ordered phase we can change a generalized force $\beta_1$ so that the free energy curve changes from blue to red. As this is done, the order parameter that minimises the free energy changes continuously. As the red curve is approached, there is a divergence of fluctuations of the order parameter. We could also approach the continuous phase transition from the unordered phase (cyan curve). In this case, the mean of the order parameter would not change, but order parameter fluctuations would still diverge.
First order phase transitions

Figure 2.2(b) shows an example of a first order phase transition. We imagine some value of parameters \( \beta \) give rise to the blue curve and a change of one of the \( \beta \), say \( \beta_1 \), changes the free energy curve from the blue curve to the red curve. As a result, there is a change in the position of the global minimum of the free energy curve and so the system would reorder into a new phase. This gives rise to a discontinuity in the equilibrium value of the order parameter as a function of \( \beta_1 \). There is also a discontinuity in at least one of the first derivatives \( \partial F / \partial \beta_i \), and this gives rise to discontinuities in the corresponding mean values \( \langle V_i \rangle \), since \( \langle V_i \rangle = \partial F / \partial \beta_i \) (Eq. (2.6)). The reordering of the system is associated with a latent heat, which is given by the discontinuity in the mean energy.

First order phase boundaries

We now imagine passing between the blue and red curves in Fig. 2.2(b). At some point the free energy curve will resemble that shown in Fig 2.2(c). This curve corresponds to a point precisely on the first order phase boundary, where the phases either side of the transition are both stable.

Continuous phase transitions

It is also possible to change the free energy profile so that the local maximum separating the two local minima in Fig. 2.2(b) is reduced and the two minima converge. This results in a transition like from the blue to the red curve in Fig. 2.2(d). In this case, the order parameter changes continuously with \( \beta_1 \). However, the fluctuations of \( \phi \), which are given by the second derivatives \( \partial^2 F / \partial^2 \beta_1 \) (Eq. (2.7)), now diverge. This is a continuous phase transition. We can also approach the continous phase transition from an unordered \( \langle \tilde{\phi} = 0 \rangle \) phase like that arising from the cyan curve in (d). In this case, the mean of the order parameter doesn’t change as one approaches the transition, but fluctuations of the order parameter diverge. Points in a phase diagram where there is a continuous phase transition are also known as critical points.

To summarise, phase transitions arise from nonanalytic changes in the properties of the global minimum of the free energy. For example a new global minimum may form (a first order phase transition) or two minima may converge (a continuous phase transition). Note that it is the equilibrium properties of the system that undergo abrupt changes across a phase transition,
while the dynamics that drive the system evolution will still be continuous. In fact, the dynamics may occur over a considerable time scale, for example the freezing of water can take a considerable amount of time and may not even occur if the liquid is unperturbed, in which case the water remains out of equilibrium in a metastable *supercooled* state. The results in this thesis will be concerned with how a system evolves toward equilibrium when quenched across a continuous phase transition. Scale invariance that emerges in the system then allows for a simple description of certain properties of the system. We will now discuss this idea of scale invariance arising from a critical point, and some of the simple properties that result.

### 2.4 Scale invariance at a critical point

A general order parameter field $\phi(x)$ in equilibrium can be decomposed into some statistical average field $\langle \phi \rangle$ and fluctuations $\delta \phi$ that depend on space and time. For thermal systems, the probability of a fluctuation $\delta \phi$ is (see Eq. (2.8)),

$$P_{\delta \phi} = \frac{\exp (-H (\langle \phi \rangle + \delta \phi))}{Z},$$

where the partition function $Z$ is the normalization factor (see Eq. (2.10))

$$Z \equiv \sum_{\delta \phi} \exp (-H (\langle \phi \rangle + \delta \phi)).$$

(2.13)

The fluctuations $\delta \phi$ affect correlations of $\phi$. The connected correlation function of $\phi$ between two spacetime$^6$ points $(x_1, t_1)$ and $(x_2, t_2)$ is defined as,

$$G(x_1, t_1, x_2, t_2) \equiv \langle \phi(x_1, t_1) \cdot \phi(x_2, t_2) \rangle - \langle \phi(x_1, t_1) \rangle \langle \phi(x_2, t_2) \rangle$$

$$= \langle \delta \phi(x_1, t_1) \cdot \delta \phi(x_2, t_2) \rangle,$$

(2.14)

where the second line follows from the fact that $\langle \delta \phi \rangle = 0$, by construction. The average is a statistical average. The dot product ‘$\cdot$’ is understood as an appropriate scalar product valid for elements of the order parameter space. In the cases of interest for this thesis, this dot product will be either scalar multiplication or the usual vector dot product. For higher rank tensors the dot product must be generalized, for example using the Frobenius inner product.

$^6$We borrow this term from the theory of relativity, although at no point will we extend the Galilean invariant theory to a Lorentz invariant theory i.e. we are not considering any relativistic corrections. The term “spacetime” is just a convenient way to incorporate both the space and time coordinates into a single term.
for rank-2 tensors, but such results will not appear in this thesis. We will assume the system of interest is invariant under spacetime translations, so that we can set \( G(\mathbf{x}_1, t_1, \mathbf{x}_2, t_2) \rightarrow G(\mathbf{x}_2 - \mathbf{x}_1, t_2 - t_1) \) in Eq. (2.14).

Far away from a critical point, the correlation function is an analytic function of spacetime. Often there is an exponential decay of correlations \([311, 312]\),

\[
G \sim \exp \left( -\frac{|\mathbf{x}|}{\xi} \right) \quad \text{and} \quad G \sim \exp \left( -\frac{t}{\tau} \right),
\]

(2.15)

where \( \xi \) is the correlation length and \( \tau \) the correlation time. We have assumed isotropy in space by using a single correlation length \( \xi \). Both \( \xi \) and \( \tau \) are functions of the generalized forces \( \beta_i \). As one approaches a critical point, fluctuations of the order parameter diverge and so the correlation length and correlation time diverge. For the case of argument, we imagine that the critical point occurs at the point \( \beta_i = 0 \) and we take \( \beta_i > 1 = 0 \) and let \( |\beta_1| \rightarrow 0 \).

The correlation length and correlation time then diverge as power laws,

\[
\xi \sim |\beta_1|^{-\nu},
\]

\[
\tau \sim |\beta_1|^{-z\nu},
\]

(2.16)

for exponents \( \nu \) and \( z \) called critical exponents.

Quantities that depend on the strength of fluctuations in the system, for example heat capacity, also diverge as a power law. The exponent \( z \) is termed the dynamic critical exponent and will be central to this thesis. From Eq. (2.16) we see that \( z \) is a measure of the anisotropy between space and time. If we were to allow for additional anisotropy in space then this could lead to different divergences of the correlation lengths along different directions.

Precisely at the critical point, for sufficiently large \( |\mathbf{x}| \) or \( t \), the correlation function decays as a power law rather than exponentially \([311, 313, 314]\),

\[
G \sim |\mathbf{x}|^{-\eta} \quad \text{and} \quad G \sim t^{-\eta/z},
\]

(2.17)

for an additional critical exponent \( \eta \). Again, \( z \) reflects the anisotropy between space and time. In this case the inverse of \( z \) appears since a large divergence in correlation time results in a flatter (smaller \( \eta/z \)) correlation function.

The fact that the functions \( \xi(\beta_1), \tau(\beta_1) \) and \( G(\mathbf{x}, t) \) behave as power laws close to or at a critical point is of great utility as it implies scale invariance. By this we mean that the shape of a function \( f(u) \sim u^{-\alpha} \) does not depend on the scale we use to measure \( u \). To see this, note that \([315]\),

\[
f(au) \sim a^{-\alpha}u^{-\alpha} \sim a^{-\alpha}f(u),
\]

(2.18)
so that the scale factor $a$ scales the height of the function but otherwise leaves the shape of the function unchanged. The physics that gives rise to a power law must therefore be independent of length scale, and so does not depend on microscopic details of the system! This is an exciting prospect. Microscopic details of a system are in general complicated, with a huge number of degrees of freedom. Observables that are independent of microscopic details must depend on more general properties of the system, which allows for great simplifications. In the case of continuous phase transitions, these general properties are the system dimension, conservation laws, and order parameter symmetries and dimension. Microscopically distinct systems that share these general properties display the same scale invariant behaviour close to a critical point. We say that such systems belong to the same universality class. A universality class is parameterized by a small number of critical exponents that describe the power law divergences of the transition. A monumental achievement in the history of continuous phase transitions was the discovery of a theoretical basis for this scale invariant and universal behaviour. The theory is based on the renormalisation group and will be discussed in Sec. 2.7.

\section{Quantum phase transitions}

Our discussion of phase transitions so far has focussed on thermal phase transitions. A different type of phase transition can emerge if we let the temperature, and therefore also the entropy, of a system go to zero.\footnote{Note the special role played by temperature here; see Footnote 1 at the start of this chapter.} The free energy is then given simply by the energy $E$ of the system and in equilibrium the system will be in a ground state. Imagine now that the ground state energy is a function of some external parameter $g$ i.e. $E = E(g)$. Quantum phase transitions occur at points $g = g_0$ where $E(g)$ becomes nonanalytic. The phase transitions mainly considered in this thesis will be quantum phase transitions.

The symmetry breaking that occurs in quantum phase transitions is driven by quantum fluctuations.\footnote{Unlike in a classical system, a quantum system can exist in a superposition of symmetry breaking ground states. Such a macroscopic superposition can in principle result in a state with no symmetry breaking. In practice, a single symmetry breaking state will be favoured from small symmetry breaking perturbations, such as an infinitesimal external field \cite{239}. Also, tunneling between symmetry breaking states is lost in the thermodynamic limit \cite{239,316}.} Compare thermal phase transitions, where the symmetry breaking that occurs is driven by thermal fluctuations. Thermal fluctuations are zero at zero temperature, however quantum fluctuations are...
still present.

The notion of first order and continuous phase transitions can be applied to quantum phase transitions, although the terms require elaboration. One approach could be to follow the definitions used in thermal phase transitions and consider derivatives of the free energy with respect to external parameters. Since the free energy is just the energy at zero temperature, first order phase transitions would arise from a discontinuity in the first derivative of the ground state energy, whereas continuous phase transitions would have a discontinuity at some higher derivative. The problem with this interpretation is that, unlike in the classical case, fluctuations are not determined by derivatives of the free energy. The most important distinction for us between first order and continuous phase transitions is whether or not fluctuations diverge at the transition, as this determines whether or not the system exhibits scale invariance. We would therefore like to invoke a distinction that captures this important difference.

In [239], Sachdev defines a continuous quantum phase transition “loosely” as one where the energy scale of fluctuations above the ground state vanish as one approaches a critical point \( g_0 \). The relevant fluctuations are those that affect the symmetry of the system. If such fluctuations lie at a frequency \( \Delta \) above the ground state, then as we approach the critical point the energy scale often vanishes as a power law [239],

\[
\Delta \sim |g - g_0|^{2\nu}.
\]

As a result, correlation lengths and time diverge as in Eq. (2.16). Therefore, although the origin of fluctuations is different between quantum and thermal phase transitions, the nature of divergences at a continuous phase transition is the same. In fact, a quantum phase transition may continuously transform into a thermal one if the temperature is increased to the point where thermal fluctuations wash out quantum fluctuations. The similarities between thermal and quantum continuous phase transitions allow us to apply many of the same scaling ideas to both, in particular the ideas of scale invariance in phase ordering dynamics that will be introduced in the next section.

\section{The theory of phase ordering dynamics}

\subsection{Origin of phase ordering dynamics}

The theory of phase ordering dynamics describes how the order in a system evolves following a quench across a continuous phase transition. An archetypal example of such a quench is shown in Fig. 2.3. The free energy plots in Fig. 2.3
Figure 2.3: An archetypal example of a quench leading to phase ordering dynamics. The system begins in equilibrium in (a) and then a quench drives the free energy curve to (b) so that the system is out of equilibrium. Local fluctuations in the system drive a local choice of $\phi(x)$ from one of the two wells in (b). This is the initial stage of phase ordering, and results in local order being established across some microscopic length scale $\xi$ over some microscopic time $\tau$. These domains of local order will grow and compete for the global equilibrium state. This growth of order parameter domains is the second stage of phase ordering.

could describe, for example, Ising ferromagnetism in zero external magnetic field. In this case, Fig. 2.3(a) would correspond to the high temperature, unmagnetised phase, and Fig. 2.3(b) to the low temperature, magnetized phase. Figure 2.3 then describes a quench from the unmagnetized phase to the magnetized phase. The system is assumed to be in equilibrium in Fig. 2.3(a) (i.e. $\phi(x) = 0$), so that immediately following the quench the system is out of equilibrium. The theory of phase ordering dynamics describes how such an out of equilibrium system evolves toward equilibrium.

The process of phase ordering dynamics can be broken into two stages: an initial evolution that results in local order and a long time evolution that results in global order. We can understand the first stage using a simple picture of a marble rolling in the free energy curves Fig. 2.3. The position of the marble represents the local order. Before the quench, the system is unmagnetised and the marble lies at the bottom of the free energy curve in Fig. 2.3(a). An instantaneous quench changes the free energy profile while leaving the system undisturbed. Therefore the marble is forced to the top of the hill in Fig. 2.3(b), where it is in unstable equilibrium. For the system to magnetize in some local region, the marble must roll down the hill. In zero
external magnetic field, the marble has a choice which side to roll down. If there are exactly zero perturbations then the marble will be in an unstable equilibrium at the top of the hill. In reality, there will be fluctuations, such as noise, thermal fluctuations or, even in a perfectly clean system, quantum fluctuations. These fluctuations create local conditions that force the marble down one side of the hill. The marble rolling down the hill represents the local growth of order and corresponds to a local breaking of symmetry. This is the first stage of phase ordering dynamics.

In most cases, the fluctuations driving the local breaking of symmetry will be uncorrelated across space, and therefore order will be chosen differently in different regions of the system. We represent this spatially dependent ordering of the system by an order parameter field $\phi(x)$. Correlations in the field $\phi(x)$ will occur across length scales over which the system may communicate as it locally orders. Roughly speaking, if the marble takes time $\tau$ to roll down the hill in Fig. 2.3 and excitations in the system travel at a speed $c$ then initial correlations of $\phi(x)$ will be of size $\xi \sim c\tau$. The length and time scales $\xi$ and $\tau$ will be determined by the microscopic details of the system.$^9$

A system with just local order will not be in equilibrium, since the presence of many small domains of size $\xi$ costs excessive energy. The system continues to evolve toward an equilibrium state with long (or quasi-long) range order. A long range ordered state has correlations of $\phi(x)$ that extend right across the system. A quasi-long range ordered state has correlations of $\phi(x)$ that are large, but do not reach right across a system in the limit of infinite system size.$^{10}$ This evolution from a locally ordered phase to a globally ordered phase describes the second stage of phase ordering dynamics, see Fig. 1.1. Throughout this stage, we identify some domain size that describes the length

$^9$More precisely, the process can be understood as follows. The correlations that develop in the system shortly after the quench arise from a superposition of unstable modes as $G(r) = \sum_k \exp(ik \cdot r + \omega(k)t)$ (note that the modes grow exponentially with time rather than oscillate). At $t = 0$ these give rise to delta correlations $G(r) = \delta(r)$. As time evolves, the amplitudes of waves with different “spectrums” $\omega(k)$ grow at different rates and this leads to an accumulation of wavevectors around $k_f$, the fastest growing mode. After a time

$$\tau \sim \left(\frac{k_f^2 d^2\omega}{dk^2}\right)^{-1/2}$$

the amplitudes will have separated such that correlations will be dominated by the wavevector $k_f$ (if the second derivative of the dispersion relation evaluates to zero then one would look at the next highest even derivative and so on; the first odd derivative is necessarily zero for a maximum to occur at $k_f$).

$^{10}$More precisely, the correlations decay algebraically with some exponent that is a monotonically increasing function of temperature.
scale across which the system is ordered. This begins as $\xi$ and then grows. The growing domains compete for the global equilibrium state.

The initial stage of phase ordering dynamics can be understood from the microscopic model of the system. We imagine some dominant initial phase and then carry out a linearized mode expansion around this phase to identify the unstable modes that will grow following the quench. The nature of these modes determines the microscopic quantities $\xi$ and $\tau$. Once the population of unstable modes becomes large, a mode expansion is no longer valid. To continue with a microscopic description of the system would require modelling complex excitations involving multiple length scales and interactions. For most large systems this makes microscopic calculations impractical. Fortunately, a relatively simple understanding of the second stage of phase ordering dynamics can be obtained from the notion of \textit{scale invariance}. This gives rise to the dynamic scaling hypothesis that we will now discuss.

### 2.6.2 The dynamic scaling hypothesis

The concept of scale invariance has already been introduced in Sec. 2.4 in the discussion of continuous phase transitions. Perhaps somewhat surprisingly, similar ideas can be applied to the nonequilibrium phase ordering dynamics following a quench \textit{across} a continuous phase transition. We introduce a single time spatial correlation function that describes the growing order in the system,

$$G(x_1, x_2, t) = \langle \phi(x_1, t) \cdot \phi(x_2, t) \rangle,$$  

(2.21)

where $x_1$ and $x_2$ are two spatial points and angular brackets denote a statistical average. The statistical average is over evolution trajectories that differ due to noisy initial conditions and/or noisy evolution. We assume translational invariance so that $G(x_1, x_2, t)$ can be written as $G(x_1 - x_2, t)$. This allows us to perform a spatial average as well as any ensemble average, so that we can write,

$$G(x, t) = \frac{1}{l^D} \int d^D\mathbf{x}' \langle \phi(x, t)\phi(x + x', t) \rangle,$$  

(2.22)

where $l$ is the system size and $D$ the system dimension. Assuming rotational symmetry, we can also average over radial shells and write the correlation function as $G(r, t)$ with $r = |x|$. The \textit{dynamic scaling hypothesis} says that when correlations of the order parameter are much larger than the microscopic length scale $\xi$, the growth of correlations becomes scale invariant. By this we
mean that a system at later times looks statistically identical to a system at earlier times, but with rescaled lengths: in essence, evolving forward in time has the same effect as if we stretched the system in space, see Fig. 1.1.

Quantitatively, this scale invariance means that if we scale the spatial coordinate of the correlation function by some growing macroscopic length scale \( L(t) \), the correlation functions at different times collapse onto a single function \( f \),

\[
G(r, t) = f \left( \frac{r}{L(t)} \right). \tag{2.23}
\]

The length scale \( L(t) \) is a measure of the size of the growing order parameter domains. In addition to the result (2.23), the growing length scale \( L(t) \) is predicted to grow as \( t^{1/z} \) where \( z \) is the dynamic critical exponent for the system, introduced in Sec. 2.4. The scaling result (2.23) then takes the simple form,

\[
G(r, t) = f \left( \frac{r}{t^{1/z}} \right). \tag{2.24}
\]

Equation (2.24) is the quantitative form of the dynamic scaling hypothesis. This is an hypothesis on how a system will evolve from a locally ordered state to a globally ordered state following a quench across a continuous phase transition. Many results in this thesis are applications of the hypothesis (2.24) to quenches in ferromagnetic BECs, and so this prediction of how order parameter domains grow is very important to this thesis.

### 2.6.3 Topological defects

A powerful addition to understanding phase ordering dynamics is the notion of topological defects. Topological defects are stable excitations that may arise in a system from spatial variation of the order parameter \( \phi(x) \). To illustrate this, we consider the quench depicted in Fig. 2.4. This is a generalization of Fig. 2.3 to cases where the order parameter field is a vector field \( \phi(x) = (\phi_x(x), \phi_y(x)) \). Fluctuations in the system may then give rise to a vector field \( \phi(x) \) with features that look like Fig. 2.5. Such a feature is called a vortex. Around the vortex centre there is a rotation of the order parameter,

\[
\oint_C \nabla \cdot \nabla \text{Arg} (\phi_x + i\phi_y) = 2\pi \kappa, \quad \kappa \in \mathbb{Z} \setminus \{0\}, \tag{2.25}
\]

where \( C \) is a loop enclosing the vortex centre. The parameter \( \kappa \) is the vortex charge and is forced to be integer so that the order parameter is single
valued at the start and end points of $C$. The fact that the vortex charge is an integer means that a vortex cannot be removed from continuous time evolution, termed *topological protection*. To see this, we imagine a system with a vortex (2.25) and evolve the system an infinitesimal step forward in time. As a result the vortex charge will either not change or change infinitesimally. But since the vortex charge must be an integer, it cannot change infinitesimally so is an invariant.

The vortex charge will not change if we make the loop $C$ smaller and smaller, and so to avoid a divergence in spatial derivatives of $\phi$ the magnitude of the order parameter must be zero at the vortex centre. In ordered systems $\phi(x) = 0$ does not lie on the order parameter manifold, and so the vortex core is a *defect* in the order parameter field. Since the defect arises from the topology (phase winding) of the vortex, a vortex is a *topological defect*.

Other topological defects are also possible, such as solitons, domain walls, skyrmions, monopoles, hedgehogs, etc.. The common property of a topological defect is an invariant charge arising from an integral over a closed loop, or more general closed surface. This charge, arising from the topology of the order parameter manifold, gives rise to a defect in the order parameter field $\phi(x)$.\(^\text{11}\)

The production and dynamics of defects is intimately linked with phase

\(^{11}\)More formally, the set of points $\phi(x)$ forms some topological space. Now consider all closed loops within this topological space that pass through a chosen point. Some of the loops can be continuously transformed into each other while remaining within the space, and these form an *equivalence class*. Topological defects correspond to holes in the space so that not all loops can be continuously transformed into each other. The set of equivalence
Figure 2.5: Schematic of a vortex topological defect. The black cross marks the centre of the defect, around which there is a unit circulation of the order parameter $\phi = (\phi_x, \phi_y)$, see Eq. (2.25).
ordering dynamics. A sufficiently noisy initial state will contain many topo-
logical defects. The length scale across which the system is ordered will then
be limited by the distance between defects, as the order parameter variation
around a defect destroys order. A single defect is topologically protected.
However, a loop around two defects of equal and opposite charge has no net
charge and therefore the defects are able to collide and annihilate. When
defect annihilation drives order formation, the defect density will decay in
a way consistent with scale invariance. In particular, the defect density will
decay as $L(t)^{-D} \sim t^{-D/z}$ where $D$ is the dimension of the system. If the
system has no net topological charge then all defects are able to annihilate
and only then can the system be in a globally ordered state.

The topological defects that are permissible in a system depends on the
nature of the order parameter and the dimension of the system. This provides
a connection between the general properties of a system that determine the
dynamic critical behaviour and the defects supported. In some systems the
non analytic properties of a phase transition is governed by changes in the
stability of topological defects. In these topological phase transitions the
defects play a role in not only the phase ordering dynamics but also in the
equilibrium properties of the transition. For the simplest order parameters,
identifying permissible defects is not too difficult. However, as the complexity
of the order parameter increases, identifying topological defects can become
challenging, and can constitute a major part of studying phase transitions.\textsuperscript{12}

\section{The renormalizataion group formulation of critical phenomena}

In this section we discuss the renormalization group (RG) formulation of
critical phenomena. From this we will derive the dynamic scaling hypothesis

\textsuperscript{12}A good example of a seemingly simple system with topologies that have proved
challenging to study is the classical Heisenberg model in two and higher dimensions [320,
321].
from Sec. 2.6. This section provides a deeper insight into the theory of critical phenomena, and in particular the theory of dynamic critical phenomena, which is the theme of this thesis. We wish to point out, however, that the discussion of dynamic scale invariance presented in Sec. 2.6 is sufficient to understand the results in this thesis, and we will not refer to this present section in our results (apart from one brief footnote). We firstly introduce the ideas of the RG procedure with a static system. The analysis we give is a somewhat standard analysis found in textbooks [311, 315, 322, 323]. After introducing the RG procedure we will generalise to dynamic systems.

2.7.1 Coarse-graining and the renormalization group procedure

Figure 2.6(a)(i) shows a two dimensional Ising system in a representative microstate in equilibrium at the critical point. We can do a coarse-graining of the image by mapping the image onto a coarser grid. To make the image coarser by a factor of \( m \), we assign to each square of \( m \times m \) lattice points a single spin, determined in some way from the \( m^2 \) original spins in the block. For example, we could take the spin that has the largest majority, or use the centre spin, or use some other means. The coarser lattice obtained from such a rescaling is shown in Fig. 2.6(a)(ii), where the spin with the largest majority is selected from blocks of size \( 3 \times 3 \). The essential and defining feature of the critical point is that the coarsened image is statistically similar to the original one. By this we mean that the new image is in a configuration that has the same probability as the original microstate. The entire theory of critical phenomena and the RG procedure rests on this feature. That is, at the critical point the statistical properties of the system are invariant under a RG transformation.

We can make this statement of scale invariance quantitative as follows. The statistical properties of our original lattice \( L_0 \) are contained in the partition function,

\[
Z_0 = \sum_{S_0} e^{-\mathcal{H}_0(S_0)}, \tag{2.26}
\]

where we absorb the temperature dependence into the reduced Hamiltonian \( \mathcal{H}_0 \) and \( S_0 \equiv \{s_i^0\} \) denotes a particular configuration of lattice spins \( s_i^0 \) at lattice points \( i \). The expectation value of some quantity \( A(S_0) \) is given by,

\[
\langle A \rangle = \frac{1}{Z_0} \sum_{S_0} e^{-\mathcal{H}_0(S_0)} A(S_0) . \tag{2.27}
\]
Figure 2.6: The effect of a RG transformation on an Ising system for different temperatures. Black squares represent one spin direction and white squares represent the opposite spin. The labels (i)-(v) denote successive RG transformations. (a) At the critical temperature $T_c$. The renormalization does not change the statistics of the spin distribution after each renormalization: spin correlations remain across the entire system size. (b) For temperatures higher than $T_c$ the renormalization procedure drives the system towards the infinite temperature fixed point. (c) For temperatures lower than $T_c$ the renormalization drives the system towards the zero temperature fixed point. Note that the lattice is rescaled after each transformation so that (ii) covers more of the system than (i), and (iii) covers more of the system than (ii). Figure adapted from [324].
We can carry out a RG transformation (i.e. a coarse-graining transformation) whereby we obtain a new, coarser lattice \( L_1 \) with a spin configuration \( S_1 \equiv \{ s_i^1 \} \) that is entirely determined by the partition function \( Z_0 \). We can introduce a new, arbitrarily complicated Hamiltonian \( \mathcal{H}_1(S_1) \) such that the partition function satisfies,

\[
Z_0 = Z_1 \equiv \sum_{S_1} e^{-\mathcal{H}_1(S_1)}.
\]

What do we mean by Eq. (2.28)? The partition function \( Z_0 \) is a function of parameters in the Hamiltonian \( \mathcal{H}_0 \) (it is not a function of the microstates \( S_0 \)!). The partition function \( Z_1 \) is a function of new renormalized parameters that appear in \( \mathcal{H}_1 \) from the coarse-graining transformation. The renormalized parameters in \( \mathcal{H}_1 \) are functions of the original parameters in \( \mathcal{H}_0 \), and are chosen such that the statistics of observables obtainable from \( Z_1 \) are unchanged under the transformation. This is what we mean by the equality in Eq. (2.28). What observables are obtainable from the partition function \( Z_1 \)? Defining the lattice spacing of \( L_0 \) by \( b_0 \) and \( L_1 \) by \( b_1 \), the partition function \( Z_1 \) contains information only about observables with length scales \( b_1 \) or greater; information about observables with length scales less than \( b_1 \) has been lost. But since we are interested in large length scale properties of a system when we study critical phenomena, we do not lose crucial information by this transformation.

We can continue this process indefinitely, moving to coarser and coarser grids \( L_n \) with larger and larger lattice spacings \( b_n \), but with a partition function \( Z_n \) that gives the same statistics for quantities with length scales \( b_n \) or greater. This renormalisation process is entirely defined by the flow

\[
\mathcal{H}_0 \rightarrow \mathcal{H}_1 \rightarrow ... \rightarrow \mathcal{H}_n.
\]

To quantify this flow, we consider a completely arbitrary Hamiltonian that includes all possible couplings \( j \) between different spins and between spins and external fields. We denote the strength of coupling \( j \) by \( K_j \), which map out some parameter space with coordinate axes \( \hat{K}_j \), as shown in Fig. 2.7. The Hamiltonian \( \mathcal{H}_0 \) can be defined by a vector \( \mathbf{K}_0 \equiv (K_0^1, K_0^2, ...) \) that resides in this parameter space. Many of the couplings in \( \mathcal{H}_0 \) will be zero but they may become nonzero under a RG transformation \( R(\mathbf{K}_0) = \mathbf{K}_1 \). The RG flow (2.29) can be analysed by considering paths through the parameter space, see Fig. 2.7. This alternative way of conceptualising a Hamiltonian, i.e. considering the Hamiltonian as a function of possible couplings, is crucial to the RG formulation.

\[\textsuperscript{13}\text{This transformation is not necessarily invertible, and so the transformations form a semigroup rather than a group [323].}\]
Figure 2.7: Schematic of a possible RG flow through a Hamiltonian parameter space with three couplings $j = 1, 2, 3$ and corresponding coordinate axes $\hat{K}_1, \hat{K}_2, \hat{K}_3$. 
In general, the RG process does not rely on the presence of a lattice and can be formulated for continuous systems. In this case, we would write down a Hamiltonian that is an integral over all possible products of the order parameter and its derivatives, and couplings of the order parameter to external fields, consistent with the symmetries and conservation laws of the system. The parameter space would consist of each of the prefactors of the infinite number of these terms. Note the essential role played here by the conservation laws and symmetries of the system: these, along with the dimension of the system, determine entirely the parameter space. In a continuous system we would generalise the lattice $L_n$ to a continuous coordinate system and the lattice spacings $b_n$ to a convenient length scale of this coordinate system. In a continuous system the partition function would take the form,

$$Z = \int d[\phi] e^{-\mathcal{H}(\phi)},$$

where the integral is a functional integral over all possible order parameter configurations. Equation (2.27) then generalises to,

$$\langle A \rangle = \frac{1}{Z} \int d[\phi] e^{-\mathcal{H}(\phi)} A(\phi).$$

### 2.7.2 Critical points

Interesting effects arise when the RG transformation

$$R(K_n) = K_{n+1}$$

has a fixed point. A fixed point is defined as a point where the transformation $R$ leaves the vector $K_n$ invariant, that is,

$$R(K_n) = K_n \equiv K^*.$$  

This means that the statistics at length scales $b_n$ are the same as the statistics at length scales $b_{n+1}$. In particular, if we measure lengths for each coordinate system $L_n$ in units of the ($n$ dependent) length scale $b_n$ then the correlation length of $L_n$ and $L_{n+1}$, say $\xi_n$ and $\xi_{n+1}$ respectively, are the same,

$$\xi_n = \xi_{n+1}.$$  

Whether we are at a fixed point or not, we also have $Z_{n+1} = Z_n$ (see Eq. (2.28)). Therefore both partition functions will also give the same correlation length $\xi$ when measured in some fixed unit $a \geq b_{n+1}$. Combining this with Eq. (2.34) gives $\xi = \xi_n a/b_n = \xi_{n+1} a/b_{n+1}$. Since $b_{n+1} > b_n$, this equality is only possible if $\xi = 0$ or $\xi = \infty$. 

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The case $\xi = 0$ corresponds to a flow toward an infinite temperature system. In this case, no large length scale properies survive under a RG transformation and we regard the system as being unordered. The case $\xi = \infty$ may correspond to a flow toward a zero temperature system. In this case, large length scale properties are present due to the order in the system. Both of these fixed points are stable in the sense that all points close to the fixed point are attracted to the fixed point under a RG transformation. This can be argued as follows. For a high temperature system, the system will have a finite correlation length and so the coarse graining process will reduce the correlations between lattice points and so increase disorder in the system [315]. The system is therefore attracted to the infinite temperature fixed point. In addition to the rescaling of the lattice there is also a renormalization of terms in the Hamiltonian. For rescaling by a factor of $m$, the coupling between lattice points increases by a factor of $m^{D-1}$ where $D$ is the dimension of the system. For $D > 1$, this rescaling favours the formation of order and at sufficiently low temperature this will dominate over the increase in lattice spacing, resulting in an attraction to the zero temperature fixed point. Therefore the zero temperature fixed point is also stable [311]. Separating these two fixed points is the more interesting case of an unstable fixed point, at which $\xi = \infty$. Unstable fixed points are termed critical points and give rise to the scale invariant behaviour that is observed in continuous phase transitions.\footnote{Note the role played by the dimension of the system in this argument. In one dimension only the infinite temperature fixed point is stable, so that all Hamiltonians with non-zero temperature are attracted to this point. This explains the absence of order in the Ising model in one dimension at any non-zero temperature.}

Figure 2.6(a)-(c) shows an example of successive RG transformations applied to a two dimensional Ising system. For a temperature slightly above or below the critical temperature $T_c$, the renormalisation drives the system toward an infinite or zero temperature state respectively. When the system is at a temperature $T_c$, the renormalisation does not change the system statistics and fluctuations persist after each renormalisation.

2.7.3 Universality and scaling laws

The stability of a fixed point can be determined by linearising the transformation $R$ about the fixed point,

$$K_i \approx K_i^* + \sum_j \frac{\partial R_i}{\partial K_j} \bigg|_{K^*} (K_j - K_j^*),$$

\hspace{1cm} (2.35)

\footnote{If we looked at the correlation length of fluctuations about the mean value of the order parameter, this would go to zero as we approached both the zero temperature fixed point and the infinite temperature fixed point but not a critical point.}
where \( R_i(K_n) = K_i^{n+1} \). We then examine the eigenvectors and eigenvalues of the matrix \( T_{ij} \) defined by,

\[
T_{ij} \equiv \left. \frac{\partial R_i}{\partial K_j} \right|_{K^*}.
\]

(2.36)

We denote the eigenvectors of this transformation by \( u_\mu \) and the corresponding eigenvalues by \( \lambda_\mu \). We can write the eigenvalues as \( \lambda_\mu = \Gamma^{a_\mu} \), where \( \Gamma \) is the factor that we rescale our system by at each renormalization step and \( a_\mu \) is some exponent independent of \( \Gamma \). To see why we can write the eigenvalues this way, note that two successive renormalization steps with rescaling factors \( \Gamma_1 \) and \( \Gamma_2 \) has the same affect as a single renormalization step with the rescaling factor \( \Gamma_1 \Gamma_2 \). Therefore \( \lambda_\mu(\Gamma_1)\lambda_\mu(\Gamma_2) = \lambda_\mu(\Gamma_1 \Gamma_2) \) (associativity).

Noting also that \( \lambda_\mu(1) = \lambda_\mu \) it follows that \( \lambda_\mu \) can be written in the form \( \lambda_\mu = \Gamma^{a_\mu} \) for some exponent \( a_\mu \), see Appendix B. The associative property \( R_{\Gamma_1} R_{\Gamma_2} = R_{\Gamma_1 \Gamma_2} = R_{\Gamma_2} R_{\Gamma_1} \) (where \( R_{\Gamma} \) is the RG transformation with a rescaling factor \( \Gamma \)) also shows that the transformations \( R_{\Gamma_1} \) and \( R_{\Gamma_2} \) commute and so the corresponding matrices \( T_{ij}(\Gamma_1) \) and \( T_{ij}(\Gamma_2) \) also commute. They can therefore be simultaneously diagonalized so that the eigenvectors \( u_\mu \) are independent of the choice of \( \Gamma \).

The sign of \( a_\mu \) for each eigenvalue determines the stability of the fixed point along the direction \( u_\mu \). An eigenvalue with positive \( a_\mu \) grows under a RG transformation and therefore corresponds to a direction of instability. The corresponding eigenvector \( u_\mu \) is termed “relevant”. An eigenvalue with negative \( a_\mu \) corresponds to a direction of stability. The corresponding eigenvector is termed “irrelevant”. The stability of directions \( u_\mu \) that have a corresponding eigenvalue with \( a_\mu = 0 \) cannot be determined from a linear analysis and are termed “marginal”. To study marginal eigenvectors requires expansions beyond a linear analysis, and can lead to logarithmic corrections to scaling [311]. The surface spanned by the irrelevant eigenvectors forms a critical surface. All points on this surface are attracted to the critical point under a RG transformation and therefore share the same large length scale behaviour as the critical point. The relevant eigenvectors will depend in some complicated way on the external parameters of the system. By varying these external parameters, it is possible to move along the direction of a relevant eigenvector until it intersects the critical surface. The external parameter values at the critical surface define the critical point in a system’s phase diagram. This gives rise to the concept of universality: all systems that reside in a particular parameter space will share the same critical behaviour if the appropriate external parameters can be tuned so that the systems lie on the same critical surface. We emphasise the point made earlier that a
parameter space is determined entirely by the symmetries, conservation laws and dimension of the system. A critical surface is therefore also determined by these properties and this explains the essential role these properties play in forming universality classes.

The linearisation of the RG transformation close to a fixed point (Eq. (2.35)) allows one to determine the large length scale scaling behaviour of the free energy of a system close to a critical point, as well as various quantities that can be derived from the free energy. Scale invariance manifests itself as power laws with exponents related to the eigenvalues of $T_{ij}$. The scaling behaviour of such observables close to a critical point is not of great relevance to this thesis, and so we refer the reader to more complete texts on critical phenomena for further details [311, 315, 322]. Instead, we are interested in the scale invariance of the correlation function precisely at a fixed point. For a spatial correlation function $G(r)$, scale invariance under a RG transformation gives,

$$G(r) = A G \left( \frac{r}{L} \right),$$

(2.37)

where $L$ is an arbitrary scaling factor. The prefactor $A$ will depend on the scaling factor $L$, but not the spatial coordinate $r$. We are actually interested in the extension of this result to dynamic situations. To introduce scaling in a dynamic setting, we firstly show how critical points arise in a dynamic field theory. We then derive the dynamic extension of Eq. (2.37).

### 2.7.4 A path integral formulation of a dynamic field theory

We now extend our analysis to dynamic critical phenomena. We consider order parameter configurations $\phi(s)$ that depend on the spacetime coordinate $s \equiv (x, t)$. We will find that for a large class of stochastic dynamical models, we can write expectation values of some observable $A(\phi)$ as,

$$\langle A \rangle = \int d[\Phi] e^{-S(\Phi)} A(\phi) \frac{e^{-S(\Phi)}}{\int d[\Phi] e^{-S(\Phi)}}.$$

(2.38)

Here $\Phi \equiv (\phi, \varphi)$ with $\varphi$ a “frequency” field that will be introduced, and $S(\Phi)$ is the dynamic functional (or action). This takes the same form as Eq. (2.31) if we associate $S(\Phi)$ with $\mathcal{H}(\phi)$, and therefore allows ideas of static critical

---

16To obtain actual values for the critical exponents requires much more advanced field theory calculations, see for example [315, 325].
theory to be transferred directly to the dynamic case. We now show how Eq. (2.38) arises from a stochastic dynamical model.

The dynamic model we consider is,

\[ \frac{\partial \phi}{\partial t} = F(\phi) + f(s)\eta(s), \quad \phi(x, 0) = \phi_0(x) + \chi(x). \] (2.39)

Here \( F(\phi) \) gives the non-stochastic evolution of \( \phi \), \( \eta(s) \) is a noise term coupled to \( \phi \) by \( f \), \( \phi_0 \) is a non-stochastic initial condition and \( \chi \) is a stochastic initial condition. In treatments we are aware of the \( \chi(x) \) is taken to be zero but we will allow it to be a stochastic field, as such a case is relevant to this thesis. We can incorporate the initial condition into the equation of motion by writing,

\[ \frac{\partial \phi}{\partial t} = F(\phi) + f(s)\eta(s) + \delta(t)\chi(x) \] (2.40)

with initial condition \( \phi(x, 0) = \phi_0(x) \). The \( \delta \) function in the equation of motion (2.40) introduces the stochastic part of the initial condition as an impulse at \( t = 0 \).

The expectation value of an observable \( A \) can be obtained by averaging over \( \eta \) and \( \chi \),

\[ \langle A \rangle = \int d[\eta] \int d[\chi] \int P(\eta)Q(\chi)A(\phi_{\eta,\chi}), \] (2.41)

where \( P \) is the probability distribution for \( \eta \), \( Q \) is the probability distribution for \( \chi \) and \( \phi_{\eta,\chi} \) is the solution to Eq. (2.40) for a choice of fields \( \eta \) and \( \chi \) and initial condition \( \phi(x, 0) = \phi_0(x) \).

To obtain Eq. (2.38), we follow the method in [329]. We write Eq. (2.41) as,

\[ \langle A \rangle = \int d[\phi] \int d[\eta] \int d[\chi] P(\eta)Q(\chi)\delta(\phi - \phi_{\eta,\chi})A(\phi) \]
\[ = \int d[\phi] \int d[\eta] \int d[\chi] P(\eta)Q(\chi)\delta \left( \frac{\partial \phi}{\partial t} - F(\phi) - f\eta - \delta(t)\chi \right) A(\phi), \] (2.42)

---

This stochastic model arises naturally in a classical system coupled to a thermal reservoir and with a noisy initial state [326, 327]. The model also arises in the truncated Wigner formulation for modelling condensate dynamics, in which case the initial noise arises from quantum fluctuations and the thermal reservoir is made up of modes with low occupation [328]. This latter case, albeit without the reservoir, is the case relevant to this thesis, see Secs. 3.5 and 3.11. A third possibility that gives rise to a dynamic field theory (2.38) is the path integral formulation of a full quantum problem [239, 312]. As we will work in a classical approximation of condensate dynamics (Sec. 3.5), we will not consider the full quantum dynamics.
where the $\delta$ function picks out the single field $\phi = \phi_{\eta,\chi}$ so as to obtain Eq. (2.41). This $\delta$ function is a generalisation of the usual $\delta$ function of multiple variables in the infinite variable limit. Changing the argument of the $\delta$ function will in general introduce some Jacobian determinant associated with the transformation. As discussed in [329] (and references therein), the discretisation of $\phi$ when we interpret it as a vector can be done so that this determinant is one.

In its current form, Eq. (2.42) is intractable, as the stochastic fields appear inside the argument of the delta function. We can obtain a tractable form for the functional integral by writing the $\delta$ function as the Fourier transform of an exponential. We define the Fourier transform of a functional $g(\phi)$ as,

$$\tilde{g}(\varphi) = \int d[\phi] e^{\int ds \varphi(s)\phi(s)} g(\phi),$$

(2.43)

where $i\varphi \leftrightarrow \phi$ is the Fourier variable pair. Note that we absorb $i$ into $\varphi$ in the Fourier transform so that $\varphi$ is an imaginary field. Again, this is a generalisation of the usual Fourier transform of multiple variables in the infinite variable limit. The field $\varphi$ at a particular spacetime point then corresponds to a particular “frequency” variable (multiplied by $i$), where the frequency here is related to how $g(\phi)$ varies with $\phi$.

We can now rewrite the $\delta$ function from Eq. (2.42) in the form,

$$\delta(\phi) = \int d[\varphi] e^{-\int ds \varphi(s)\phi(s)}.$$

(2.44)

Equation (2.42) can now be written as,

$$\langle A \rangle = \int d[\phi] \int d[\varphi] \exp \left(-\int ds \varphi \left(\frac{\partial \phi}{\partial t} - F(\phi)\right)\right) A(\phi)$$

$$\times \int d[\eta] \exp \left(\int ds \varphi(s) f(s) \eta(s)\right) P(\eta)$$

$$\times \int d[\chi] \exp \left(\int ds \varphi(s) \delta(t) \chi(x)\right) Q(\chi).$$

(2.45)

Progress can be made if we assume that $\eta$ and $\chi$ are delta correlated Gaussian fields, i.e. with probability distributions,

$$P(\eta) = \frac{1}{\mathcal{N}_P} \exp \left(-\frac{1}{2\sigma_P^2} \int ds \eta(s)^2\right),$$

(2.46)

18Admittedly, we may be ignoring a constant prefactor in this inverse Fourier transform. Functional integrals are notorious for not having a well defined measure, and infinite prefactors often turn up that need to be renormalised away. Our final result does not depend crucially on such constant prefactors, and so we will not concern ourselves with this sensitive issue.

19More general Gaussian correlations can be used by allowing covariance between fields at different spacetime points, but delta correlations simplify our working.
\[ Q(\chi) = \frac{1}{N_Q} \exp \left( -\frac{1}{2\sigma_Q^2} \int dx \chi(x)^2 \right), \] (2.47)

where \( \sigma_P \) and \( \sigma_Q \) are standard deviations, and \( N_P \) and \( N_Q \) are normalization factors. The functional integrals over \( \eta \) and \( \chi \) in Eq. (2.45) are then Gaussian and can then be performed (see, for example, [315]),

\[
\frac{1}{N_P} \int d[\eta] \exp \left( -\int ds \left( \frac{\eta(s)^2}{2\sigma_P^2} - \varphi(s) f(s) \eta(s) \right) \right) = \exp \left( \frac{\sigma_P^2}{2} \int ds \varphi(s)^2 f(s)^2 \right), \tag{2.48}
\]

and

\[
\frac{1}{N_Q} \int d[\chi] \exp \left( -\int dx \left( \frac{\chi(x)^2}{2\sigma_Q^2} - \varphi(x,0) \chi(x) \right) \right) = \exp \left( \frac{\sigma_Q^2}{2} \int ds \varphi(s)^2 \delta(t) \right). \tag{2.49}
\]

This gives,

\[
\langle A \rangle = \frac{\int d[\phi] \int d[\varphi] \exp (-S(\phi,\varphi)) A(\phi)}{\int d[\phi] \int d[\varphi] \exp (-S(\phi,\varphi))}, \tag{2.50}
\]

where

\[
S(\phi,\varphi) = \int ds \left[ \varphi \left( \frac{\partial \phi}{\partial t} - F(\phi) \right) - \frac{\sigma_P^2}{2} \varphi^2 f^2 - \frac{\sigma_Q^2}{2} \varphi^2 \delta(t) \right]. \tag{2.51}
\]

We have introduced a normalisation factor in Eq. (2.50) to remove any ambiguity in the normalisation of the functional integrals. The dynamic functional (2.51) is the dynamic functional appearing in Eq. (2.50). Note that the functional integral over the second field \( \varphi \) in Eq. (2.50) seems to expand the phase space of the problem. This is not really the case: the second field entered via the \( \delta \) function (2.44), and the integral over this field in Eq. (2.50) acts to select fields \( \phi \) that satisfy Eq. (2.39), and so in fact limits the phase space.

Formally, our system is now analogous to a static system, with the “partition function”

\[
Z = \int d[\phi] \int d[\varphi] e^{-S(\phi,\varphi)} = \int d[\Phi] e^{-S(\Phi)}, \tag{2.52}
\]

40
where we have introduced the two component quantity \( \Phi \equiv (\phi, \varphi) \). Note that if the field \( \phi \) is linearly coupled to an external field \( h \) in the non-stochastic evolution, the action (2.51) will contain a term \( \varphi h \). Therefore there is a conjugacy relation between \( \varphi \) and \( h \), i.e. \( \delta S/\delta h = \varphi \), which gives \( \varphi \) a physical interpretation in terms of how a system responds to the field \( h \) [329].

Equation (2.52) is the main result of this subsection. What this result tells us is that all the ideas of static critical phenomena discussed in Secs. 2.7.1-2.7.3 can now be applied to dynamic systems, using the more general partition function (2.52). Correlations of \( \phi \) are now between two space-time points \( s_i \equiv (x_i, t_i) \) and take the form,

\[
G(s_1, s_2) = \langle \phi(x_1, t_2) \cdot \phi(x_2, t_2) \rangle.
\]

(2.53)

Assuming translational invariance in space, we can write the correlation function as,

\[
G(s_1, s_2) = G(x_2 - x_1, t_1, t_2).
\]

(2.54)

2.7.5 Renormalization in the dynamic theory and the dynamic scaling hypothesis

We can proceed with the RG procedure as in the static case but allow for anisotropy between the space and time directions by using different scaling factors for lengths and time. We scale lengths by \( L \) and time by \( T \), but restrict ourselves to cases where there is only a single scaling parameter \( a \) so that \( T = T(a) \) and \( L = L(a) \). Equivalently, we can eliminate \( a \) and write \( L = L(T) \). Defining the correlation function after \( n \) renormalization steps as \( G_n \), we have that,

\[
G_{n+1}(s_{n+1}^{x_2, t_2} - s_{n+1}^{x_1, t_1}) = AG_n \left( \frac{x_{n+1}^{x_2} - x_{n+1}^{x_1}}{L(T)} \cdot \frac{t_{n+1}^{t_2}}{T}, \frac{t_{n+1}^{t_1}}{T} \right),
\]

(2.55)

where \( s_i^n \equiv (x_i^n, t_i^n) \) is a point on the renormalized coordinate system after \( n \) renormalization steps and \( A \) is a prefactor that will depend on the scaling factor \( T \). In the case of an equal time correlation function, we can suppress one of the time dependencies and write,

\[
G_{n+1}(x_{n+1}^{x_2} - x_{n+1}^{x_1}, t_{n+1}^{t_1}) = AG_n \left( \frac{x_{n+1}^{x_2} - x_{n+1}^{x_1}}{L(T)}, \frac{t_{n+1}^{t_1}}{T} \right).
\]

(2.56)

It is now simple to derive the dynamic scaling hypothesis. At a critical point, we have \( G_{n+1}(x, t) = AG_n(x, t) \). We are free to choose the scale factor \( T \).
arbitrarily and so choose $T = t_1^n / T_0$ for some arbitrary constant $T_0$. At a critical point, the correlation function is invariant under a RG transformation. In analogy with Eq. (2.37) we then have,

$$ G(x_2 - x_1, t) = AG\left(\frac{x_2 - x_1}{L(t/T_0)}, T_0\right). $$

Therefore at a critical point the correlation function depends only on the argument $(x_2 - x_1) / L(t/T_0)$. The correlation function will therefore exhibit dynamic scale invariance, as described in Sec. 2.6.

Finally, we consider what form the function $L(T)$ can take. As in the static case, if we carry out two successive renormalizations on the system, with scaling factors $T_1$ and $T_2$, this has the same effect as a single renormalization with scaling factor $T_1 T_2$. Therefore $L(T_1) L(T_2) = L(T_1 T_2)$. This is sufficient to show (see Appendix B) that $L = T^{1/z} / L_0$, where $L_0$ is an arbitrary constant and $z$ is the dynamic critical exponent. (If we choose instead to write $T$ as a function of $L$, we would obtain $T = L^z$ and so avoid the inverse exponent.) Equation (2.57) therefore becomes,

$$ G = AG\left(\frac{x_2 - x_1}{t^{1/z}}\right), $$

where for convenience we have set $T_0 = L_0 = 1$. From its definition we see that the dynamic critical exponent is a measure of the anisotropy between scaling in space versus scaling in time. This explains, for example, the connection between the divergences of the correlation length and the correlation time in Eq (2.16).

To summarise, critical phenomena arise when the parameters in a free energy (in the static case) or a dynamic functional (in the dynamic case) take values such that the statistics of the system are invariant under an RG transformation. Scale invariant properties of a system are described by power laws, and so many physical observables close to a critical point exhibit power law behaviour. In particular, a critical point in the dynamic functional gives rise to the dynamic scaling hypothesis (2.58).

It is worth making some final remarks about the generality of critical points in a dynamic field theory. While it is true that dynamic critical behaviour is present in systems at a continuous phase transition close to equilibrium, the theory presented above can give rise to more general critical points. The notion of a nonthermal fixed point, which is a critical point in a dynamic field theory, has been introduced [28–32] so that critical behaviour is not always conflated with equilibrium systems at continuous phase transitions. Even now the precise regime where dynamic critical phenomena can be applied is subject
to debate. In our work, where we quench across a continuous phase transition, we will not attempt to identify precisely the origin of the fixed point that gives rise to the dynamic critical behaviour that we observe. Most accounts assume the fixed point is the continuous phase transition, however recent work, for example work by Thomas Gasenzer, is attempting to challenge this view [33–35]. In addition, very recent work has suggested first order phase transitions can show dynamic critical behaviour with appropriate logarithmic corrections included in the RG theory [36].
Chapter 3

Background: phases in spinor Bose-Einstein condensates

3.1 The spin-$f$ Hamiltonian in zero magnetic field

In a spin-$f$ Bose-Einstein condensate, the atoms within the condensate have access to $2f + 1$ spin states. We take these states to be the projection of spin along the $z$ direction in spin space. The spin projections run in integer steps from $m = -f$ to $m = f$. We describe each spin state by a field operator,

$$\hat{\Psi}_m(x) = \sum_i \phi_i(x) \hat{a}_{m,i},$$  \hspace{1cm} (3.1)

where $\hat{a}_{m,i}$ is a destruction operator that removes a particle in the spin state $m$ from the spatial mode $\phi_i(x)$. The sum over $i$ sums over a complete orthonormal set of spatial modes $\{\phi_i(x)\}$. The field operators satisfy the commutation relation,

$$[\hat{\Psi}_m(x), \hat{\Psi}^\dagger_{m'}(x')] = \delta(x - x')\delta_{mm'}.$$  \hspace{1cm} (3.2)

We assume that the system has a low enough density that we only need to consider two-body interactions. The spin-$f$ Hamiltonian then takes the form,

$$\hat{H} = \sum_{m=-f}^{f} \int d^3x \hat{\Psi}^\dagger_m(x) \left[-\frac{\hbar^2 \nabla^2}{2M} + V(x)\right] \hat{\Psi}_m(x)$$

$$+ \frac{1}{2} \sum_{m,n,p,q=-f}^{f} \int d^3x \int d^3x' \hat{\Psi}^\dagger_q(x') \hat{\Psi}^\dagger_p(x) U_{mnpq}(x - x') \hat{\Psi}_n(x') \hat{\Psi}_m(x),$$  \hspace{1cm} (3.3)
where $M$ is the particle mass and $V(x)$ is an external potential.\textsuperscript{1} The functions $U_{mnpq}(x - x')$ describe the two-body interactions within and between different spin states. Particles in states $m$ and $n$ interact at positions $x$ and $x'$ respectively, scattering to produce particles in states $p$ and $q$ at positions $x$ and $x'$ respectively. We consider low energy $s$-wave interactions only, in which case it is permissible to replace $U$ by a delta function \textsuperscript{330},

$$U_{mnpq}(x - x') \rightarrow c_{mnpq}\delta(x - x').$$

(3.4)

In Fourier space this substitution corresponds to momentum independent scattering.\textsuperscript{2}

Conservation of total spin allows us to separate the spin interactions into spin channels, with total spin $F = 0, 1, 2, \ldots, 2f$. The exchange of two particles of spin $f$ results in the many body wavefunction acquiring a phase $(-1)^{2f}$, i.e. there is a sign change for fermions and no sign change for bosons. Alternatively we can consider separately the effect of particle exchange on the spin and relative orbital angular momentum of the two particles. An eigenfunction of the relative orbital angular momentum operator with total orbital angular momentum $l$ changes sign by $(-1)^l$ under exchange of the two particles \textsuperscript{333}, which can be seen from the parity of the spherical harmonics.

The spin component of the wavefunction acquires a phase $(-1)^{F + 2f}$ under particle exchange, which can be shown from properties of the Clebsch-Gordan coefficients \textsuperscript{334}. Setting $(-1)^l \times (-1)^{F + 2f} = (-1)^{2f}$ gives $(-1)^{F + l} = 1$. Therefore, for both bosons and fermions, $F + l$ must be even \textsuperscript{335}. For $s$-wave scattering we have $l = 0$ and so $F$ must be even. This leaves only $f + 1$ spin channels, which we label by $c_{mnpq} \rightarrow c_F = c_0, c_2, \ldots, c_{2f}$. These contact interactions are determined by the $s$-wave scattering lengths $a_F$ of the $f + 1$ spin channels,

$$c_F = \frac{4\pi\hbar^2}{M} a_F.$$

(3.5)

Spinor condensates have been realised in a number of atomic species. Experiments utilising the hyperfine levels of rubidium and sodium have produced both spin-1 \textsuperscript{240–242} and spin-2 \textsuperscript{243–246} condensates. Spin-3 condensates\textsuperscript{1}In general, an external potential may affect each spin state differently, for example in a magnetic trap. In a far detuned optical trap, however, the potential is the same for each spin state \textsuperscript{240} and we will only consider such a trap.\textsuperscript{2}Momentum independent scattering is only a good approximation for low energy scattering, and trying to apply Eq. (3.4) for all momenta leads to ultraviolet divergences. These divergences can be avoided using more careful approximations of the low energy scattering \textsuperscript{331, 332}.45
have been produced using chromium [247, 248]. The total spin of an atom is a combination of nuclear and electron spins. The simplest spinor condensate is a spin-1 system. The experimental works on spin-1 systems have used $^{87}\text{Rb}$ and $^{23}\text{Na}$ atoms. These have a nuclear spin of $3/2$ and an electron spin of $1/2$, resulting in the spin combinations $3/2 + 1/2 = 2$ and $3/2 - 1/2 = 1$. Spin-spin coupling between the nuclear and electron spin leads to a hyperfine splitting of the spin-1 and spin-2 levels, so that condensates of atoms confined to the spin-1 level can be realized.

3.2 The spin-1 Hamiltonian in zero magnetic field

In this thesis we consider a spin-1 system, the simplest spinor condensate. We will limit ourselves to studies of quasi-2D systems. Here we formulate the general three dimensional problem, indicating three dimensional quantities by a superscript “3D”. In Sec. 3.4 we show how to reduce the problem to a quasi-2D one. For a spin-1 condensate there only two spin channels 0 and 2, which can be combined to form a spin independent density-density interaction, and a spin dependent interaction. Equation (3.3) can then be written as [338, 339],

$$\hat{H} = \sum_{m=-1}^{1} \int \! d^3x \, \hat{\Psi}_m^\dagger (x) \left[ -\frac{\hbar^2 \nabla^2}{2M} + V(x) \right] \hat{\Psi}_m (x) + \frac{1}{2} \int \! d^3x \left[ g_3^{3D} : \hat{n}^{3D}(x)^2 : + g_s^{3D} : \hat{F}^{3D}(x) \cdot \hat{F}^{3D}(x) : \right],$$  \hspace{1cm} (3.6)

where $g_3^{3D} = (c_0 + 2c_2)/3$ is the density coupling, $g_s^{3D} \equiv (c_2 - c_0)/3$ is the spin coupling and $: \cdot$ denotes normal ordering. For the system to be stable, the density coupling must be positive. The spin coupling can be positive or negative, corresponding to an antiferromagnetic or ferromagnetic condensate respectively. The scattering lengths for the scattering channels in spin-1 rubidium and sodium have been measured: for $^{87}\text{Rb}$ these are $(a_B$ is the Bohr radii) $a_0/a_B = 101.8 \pm 0.2$ and $a_2/a_B = 100.4 \pm 0.1$ [340]; for $^{23}\text{Na}$ these are

---

The species listed have zero orbital angular momentum. A nonzero orbital angular momentum can combine with spin to give even higher total angular momentum, and condensates of such atoms (dysprosium and erbium) have also been realised in experiments [249, 250]. The atoms of chromium, dysprosium and erbium also have a strong dipolar interaction and this has led to a number of interesting effects being explored in experiments, such as the Rosenweig instability [162] and self-trapping condensates [336, 337].

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\[ a_0 = 47.36 \pm 0.80 \quad \text{and} \quad a_2 = 52.98 \pm 0.40 \quad [341]. \] Therefore ferromagnetic interactions occur in \(^{87}\text{Rb} \quad [340, 342, 343]\) whereas antiferromagnetic interactions occur in \(^{23}\text{Na} \quad [241, 341, 344]\). Note that since \(a_0 \approx a_2\) for both \(^{23}\text{Na}\) and \(^{87}\text{Rb}\), the density interactions are much stronger than the spin interactions, \(g_n^{3\text{D}} \gg |g_s^{3\text{D}}|\). For typical condensate densities the spin interaction energy per atom is on the order of 10Hz in \(^{87}\text{Rb} \quad [223]\) and 100Hz in \(^{23}\text{Na} \quad [241, 290]\), while the density interaction energy is a factor of 10-100 times larger.

The number density operator is,

\[
\hat{n}^{3\text{D}}(x) = \sum_{m=-1}^{1} \hat{\Psi}^\dagger_m(x) \hat{\Psi}_m(x),
\]  

(3.7)

and the spin density operator is,

\[
\hat{F}_\nu^{3\text{D}}(x) = \sum_{m,m'=\pm 1} \hat{\Psi}^\dagger_m(x) (f_\nu)_{mm'} \hat{\Psi}_{m'}(x),
\]  

(3.8)

where \(f_\nu\) are the spin-1 matrices,

\[
f_x = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad f_y = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad f_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]  

(3.9)

It can be instructive to expand the interaction parts in the Hamiltonian (3.6) in terms of the components \(\hat{\Psi}_m\).

\[ : g_n^{3\text{D}} (\hat{n}^{3\text{D}})^2 : = g_n^{3\text{D}} \left( \hat{\Psi}_1^\dagger \hat{\Psi}_1^\dagger \hat{\Psi}_1 + \hat{\Psi}_0^\dagger \hat{\Psi}_0^\dagger \hat{\Psi}_0 + \hat{\Psi}_-^\dagger \hat{\Psi}_-^\dagger \hat{\Psi}_- \right) \]

\[ + 2g_n^{3\text{D}} \left( \hat{\Psi}_1^\dagger \hat{\Psi}_0^\dagger \hat{\Psi}_0 + \hat{\Psi}_-^\dagger \hat{\Psi}_-^\dagger \hat{\Psi}_- \right) .
\]

\[ : g_s^{3\text{D}} \left| \hat{F}^{3\text{D}} \right|^2 : = g_s^{3\text{D}} \left( \hat{\Psi}_1^\dagger \hat{\Psi}_1^\dagger \hat{\Psi}_1 + \hat{\Psi}_-^\dagger \hat{\Psi}_-^\dagger \hat{\Psi}_- \right) \]

\[ + 2g_s^{3\text{D}} \left( \hat{\Psi}_1^\dagger \hat{\Psi}_0^\dagger \hat{\Psi}_0 + \hat{\Psi}_-^\dagger \hat{\Psi}_-^\dagger \hat{\Psi}_- \right) .
\]

(3.10)

The first two lines of interactions in Eq. (3.10) arise when the the two spin channels \(F = 0, 2\) have identical scattering, so that \(g_s = 0\). This interaction is identical to that of a condensate of three distinct species with equal intercomponent and intracomponent interactions. When the spin channels do not have identical scattering, so that \(g_s \neq 0\), the intercomponent and intracomponent interactions are modified depending on the total spin of the interaction. This gives rise to the scattering in lines three and four, which (when combined with
the first two lines of interactions) resembles the interactions in a condensate of three distinct species with unequal intercomponent and intracomponent interactions. In addition, \( g_s \neq 0 \) results in scattering events given by the fifth line in Eq. (3.10), which do not occur in condensates of three distinct species. This is the spin exchange interaction, and corresponds to two atoms in the \( m = 0 \) level scattering into the \( m = 1 \) and \( m = -1 \) levels, and vice versa. Such an interaction is not possible in condensates of three distinct species, since distinct species cannot be transformed into each other.

3.3 The spin-1 Hamiltonian in non-zero magnetic field

In a weak external magnetic field, which we take to point along the \( z \) direction in spin space, there is a Zeeman shift of the \( m = \pm 1 \) spin state relative to the \( m = 0 \) spin state. We will treat this perturbatively. The Zeeman shift is modified by the hyperfine splitting discussed in Sec. 3.1. For a static (i.e. DC) magnetic field \( B \), the energy shift can be evaluated using degenerate perturbation theory [333]. To first order in the magnetic field (relative to the hyperfine splitting), the Zeeman shift is,

\[
p_m \equiv -g \mu_B B m,
\]

where \( g \) is the (atom dependent) Landé hyperfine \( g \)-factor and \( \mu_B \) is the Bohr magneton. In Sec. 3.5 we will show that the precession caused by the shift (3.11) can be trivially removed, however we retain the effect for now.

The effects of hyperfine splitting on the Zeeman shift appear at second order in the magnetic field, and lead to a correction to Eq. (3.11) of [299, 333, 335, 345],

\[
q_{DC} m^2 \equiv \frac{(g \mu_B B)^2}{\Delta E_{hf}} m^2,
\]

where \( \Delta E_{hf} \) is the hyperfine splitting. A quadratic Zeeman shift can also be induced using a detuned microwave dressing field that couples the \( m = 0 \) state of the \( F = 1 \) manifold to the \( m = 0 \) state of the \( F = 2 \) manifold [283, 290, 346, 347]. The AC quadratic Zeeman shift (i.e. a Stark shift) of the \( m = 0 \) field relative to the \( m = \pm 1 \) fields is,

\[
q_{AC} \equiv -\frac{\hbar \Omega^2}{4\delta},
\]

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where $\delta$ is the detuning of the microwave field from the $m = 0, F = 1 \leftrightarrow m = 0, F = 2$ transition frequency, and $\Omega$ is the Rabi frequency. Combining the AC and DC quadratic Zeeman shifts gives a total quadratic Zeeman shift,

$$ q \equiv q_{\text{DC}} + q_{\text{AC}}. \tag{3.14} $$

The microwave dressing field can be controlled independently of the static magnetic field and therefore $q$ can be tuned independently of $p$. The microwave dressing field also allows $q$ to be tuned to both positive and negative values by changing the sign of the detuning (c.f. the DC quadratic Zeeman shift, which has a sign fixed by $\Delta E_{hf}$).

The two parts $p$ and $q$ that make up the Zeeman shift to second order in the external field are known as the linear and quadratic Zeeman shifts respectively, see Fig. 3.1. We could continue to find higher order Zeeman shifts, but we will only consider fields weak enough that second order is sufficient. We will be interested in quadratic Zeeman shifts on the order of the spin interaction energy, which for $^{87}$Rb is on the order of 10Hz, see Sec. 3.2. This can be achieved using field strengths on the order of 100mG [283]. Experiments have controlled the quadratic Zeeman energy to within a few percent of the spin interaction energy in both $^{87}$Rb [223] and $^{23}$Na [348].

The final form of the Hamiltonian in the presence of external Zeeman fields takes the form,

$$ \hat{H} = \sum_{m=-1}^{1} \int d^3x \hat{\Psi}_m^\dagger(x) \left[ -\frac{\hbar^2 \nabla^2}{2M} + V(x) + pm + qm^2 \right] \hat{\Psi}_m(x) \\
+ \frac{1}{2} \int d^3x \left[ g_{3D}^n : \hat{n}^{3D}(x)^2 : + g_{3D}^s : \hat{F}^{3D}(x) \cdot \hat{F}^{3D}(x) : \right]. \tag{3.15} $$

For zero Zeeman shifts, the Hamiltonian (3.15) has full rotational symmetry in spin space i.e. the Hamiltonian is invariant under an arbitrary spin rotation $e^{\alpha \cdot F}$ (this is a rotation by angle $|\alpha|$ about the spin direction $\alpha$). The Zeeman shifts break the spin symmetry about any axis in the $f_x - f_y$ plane but preserve the symmetry about $f_z$. Therefore with nonzero Zeeman shifts, the Hamiltonian (3.15) is in general invariant under spin rotations $e^{\alpha f_z}$ only. The ground states of Eq. (3.15) are sensitive to the Zeeman shifts. This allows the phase of a spin-1 system to be manipulated through control of the external fields. We will find that this can give rise to quantum phase transitions within a spin-1 condensate, which will be the main study of this thesis.
3.4 Quasi-2D condensate

Condensate experiments require a trapping potential $V(x)$ to confine the condensed atoms. Traps tightly confined along one spatial direction allow for quasi-2D condensates to be produced [349]. In addition, homogeneous traps in the quasi-2D [195] and 3D regimes [350, 351] have been realized. Condensates in homogeneous traps are simpler to analyse than in harmonic traps, and experiments with BECs have been moving toward such traps for studies of phase transition dynamics, for example in studies of the Kibble-Zurek mechanism across the BEC transition [195, 196]. In this work we will consider only homogeneous traps so that we can relate the physics we see to the theory of phase ordering dynamics without trap complications (see [352] for a discussion of the effects of a harmonic trap on phase ordering).

We consider a homogeneous quasi-2D trapping potential, i.e. a harmonic trap tightly confined along $z$ with no confinement along $x$ and $y$,

$$V(x) = \frac{1}{2} M \omega_z^2 z^2,$$  \hspace{1cm} (3.16)

where $\omega_z$ is the trapping frequency. By tightly confined we mean that $\hbar \omega_z$ is large compared to the energy available for each particle. This restricts the system to be in the ground state $\phi_{z,0}(z)$ of the harmonic trap, where

$$\phi_{z,0}(z) = \left( \frac{M \omega_z}{\pi \hbar} \right)^{1/4} \exp \left( - \frac{M \omega_z z^2}{2\hbar} \right).$$  \hspace{1cm} (3.17)

Figure 3.1: Schematic of the (a) linear and (b) quadratic Zeeman shifts of the $m = \pm 1$ spin states relative to the $m = 0$ spin state, for the case $p, q > 0$. 
We can then write the field operators (3.1) as,

\[ \hat{\Psi}_m(x) = \phi_{z,0}(z) \sum_i \phi_{xy,i}(x, y) \hat{b}_{m,i}, \]

where \( \{ \phi_{xy,i} \} \) is a complete set of spatial modes in the \( x-y \) plane and \( \hat{b}_{m,i} \) is a destruction operator that removes a particle in the spin state \( m \) from the spatial mode \( \phi_{xy,i} \). We can then integrate over the \( z \) direction in Eq. (3.15).

In terms of the transverse field operators \( \hat{\psi}_m \), Eq. (3.15) becomes (redefining \( x \) as \( x \equiv (x, y) \)),

\[ \hat{H} = \sum_{m=-1}^{1} \int d^2x \hat{\psi}_m^\dagger(x) \left[ -\frac{\hbar^2 \nabla^2}{2M} + pm + qm^2 \right] \hat{\psi}_m(x) + \frac{1}{2} \int d^2x \left[ g_n : \hat{n}(x)^2 : + g_s : \hat{F}(x) \cdot \hat{F}(x) : \right], \]

where

\[ g_n = g_{3D}^n \int dz |\phi_{z,0}|^4 = \sqrt{\frac{M \omega_z}{2\pi \hbar}} g_{3D}^n, \]

and

\[ g_s = g_{3D}^s \int dz |\phi_{z,0}|^4 = \sqrt{\frac{M \omega_z}{2\pi \hbar}} g_{3D}^s, \]

are effective two-dimensional couplings and

\[ \hat{n}(x) = \sum_{m=-1}^{1} \hat{\psi}_m^\dagger(x) \hat{\psi}_m(x), \]

\[ \hat{F}_\nu(x) = \sum_{m,m'=-1}^{1} \hat{\psi}_m^\dagger(x) (f_\nu)_{mm'} \hat{\psi}_{m'}(x), \]

are areal number and spin densities respectively. Note that we have ignored the inconsequential constant energy shift from the trapping potential in Eq. (3.19). Equation (3.19) is the effective two-dimensional Hamiltonian for our system. From now on we will work entirely in this two-dimensional system.

In principle, one could work in a fully quantum picture and use the Hamiltonian (3.19) to solve for the system’s dynamics. In practice, this is too difficult for most cases of interest and so we must resort to approximations. In Sec. 3.5
we describe the classical field approximation, in which we assume that the
spinor system obeys a classical equation of motion. This approximation will
be used predominantly in this thesis. In Sec. 3.6 we describe the Bogoliubov
approximation, in which we assume we have a zero momentum classical field
and allow for scattering from this field into quantum modes. The Bogoliubov
approximation serves us mainly as a means to determine the initial conditions
of our system and also provides useful analytic results to compare with the
early-time evolution of classical field simulations.

3.5 Classical field approximation

In a classical field approximation we replace the field operators \( \hat{\psi}_m \) by classical
fields \( \psi_m \) (hat removed). This approximation is valid when the populations
of the modes in \( \hat{\psi}_m \) are highly occupied and can be justified from a Wigner
formulation [328, 353, 354]. From Eq. (3.19), we obtain the Hamiltonian for
our system,

\[
H = \sum_{m=-1}^{1} \int d^2x \left( -\frac{\hbar^2 \nabla^2}{2M} + pm + qm^2 \right) \psi_m(x) \\
+ \frac{1}{2} \int d^2x \left[ g_n n(x)^2 + g_s |F(x)|^2 \right],
\]

(3.24)

where

\[
n(x) \equiv \sum_{m=-1}^{1} |\psi_m(x)|^2
\]

(3.25)
is the total density and

\[
F_\nu(x) \equiv \sum_{m,m'=\pm 1} \psi^*_m(x) (f_\nu)_{mm'} \psi_{m'}(x)
\]

(3.26)
is the spin density.

The classical fields \( \psi_m(x) \) satisfy the Poisson bracket relation,

\[
\{ \psi_m(x), \psi^*_{m'}(x') \} = \delta(x - x') \delta_{mm'}.
\]

(3.27)
The dynamics of the classical fields are given by Hamilton’s equation of
motion,

\[
i\hbar \frac{\partial \psi_m(x)}{\partial t} = \frac{\delta H}{\delta \psi^*_m(x)}
\]

(3.28)
\(\frac{\delta}{\delta f(x)}\) denotes a functional derivative) from which we obtain the spin-1 Gross-Pitaevskii equations (GPEs) [335],

\[
\mathcal{H}^m = \left( -\frac{\hbar^2 \nabla^2}{2M} + pm + qm^2 + gn \right) \psi_m + g_s \sum_{m'=-1}^{1} (\mathbf{F} \cdot \mathbf{f})_{mm'} \psi_{m'}.
\]

(3.29)

The three equations (3.29) are coupled through the density and spin interaction terms. The linear Zeeman term causes a rotation of \(\psi_{\pm 1}\), which can be removed by making the replacement, \(\psi_m \rightarrow e^{-ipm t / \hbar} \psi_m\). This gives,

\[
\mathcal{H}^m = \left( -\frac{\hbar^2 \nabla^2}{2M} + qm^2 + g_n n \right) \psi_m + g_s \sum_{m'=-1}^{1} (\mathbf{F} \cdot \mathbf{f})_{mm'} \psi_{m'}.
\]

(3.30)

Writing out the spin components of \(\mathbf{F}\) explicitly, Eq. (3.30) takes the form,

\[
\begin{align*}
\mathcal{H}^{1} &= \left( -\frac{\hbar^2 \nabla^2}{2M} + q + gn \right) \psi_1 + g_s \left( |\psi_1|^2 + |\psi_0|^2 - |\psi_{-1}|^2 \right) \psi_1 + g_s \psi_{-1}^* \psi_0^2 \\
\mathcal{H}^{0} &= \left( -\frac{\hbar^2 \nabla^2}{2M} + gn \right) \psi_0 + g_s \left( |\psi_1|^2 + |\psi_{-1}|^2 \right) \psi_0 + 2 g_s \psi_0^* \psi_1 \psi_{-1} \\
\mathcal{H}^{-1} &= \left( -\frac{\hbar^2 \nabla^2}{2M} + q + gn \right) \psi_{-1} + g_s \left( |\psi_{-1}|^2 + |\psi_0|^2 - |\psi_1|^2 \right) \psi_{-1} + g_s \psi_1^* \psi_0^2.
\end{align*}
\]

(3.31)

The interaction terms are described by and below Eq. (3.10).

The most general possible spinor for a spin-1 condensate can be parameterized as,

\[
\psi = \sqrt{n} \begin{pmatrix} \zeta_1 \\ \zeta_0 \\ \zeta_{-1} \end{pmatrix},
\]

(3.32)

\(^4\)The Gross-Pitaevskii description of Bose-Einstein condensates is a commonly used theoretical tool for both scalar and multicomponent condensates. It is often believed that this theory describes only the zero temperature condensate evolution. The theory is in fact more general than this, and can be applied to any highly occupied modes in the system, whether these are part of the condensate or not [328]. The Penrose-Onsager criterion [355] defines Bose-Einstein condensation as occurring when the largest eigenvalue of the single particle density matrix is extensive rather than intensive. The corresponding eigenstate will then be populated by a non-zero fraction of particles in the thermodynamic limit, and this state is identified as the condensate mode. This formulation applies to both the classical field \(\psi(x)\) or the original quantum field \(\hat{\psi}(x)\).
where $\zeta_m$ are complex numbers satisfying $|\zeta_1|^2 + |\zeta_0|^2 + |\zeta_{-1}|^2 = 1$. As was noted in Sec. 3.2, density interactions are much stronger than spin interactions in the most familiar spin-1 cases of $^{87}\text{Rb}$ and $^{23}\text{Na}$. For sufficiently low temperatures, fluctuations in total density will be frozen out while spin fluctuations are retained. In such a case, the manifold mapped out by the general spinor (3.32) is SU(3). In Sec. 3.7 we will consider particular submanifolds of (3.32) that are ground state manifolds for particular spin interactions and Zeeman energies. The phase ordering dynamics that we will explore in this thesis will arise from quenches between these different ground state manifolds.

### 3.6 The Bogoliubov approximation

In the Bogoliubov approximation we consider a limited range of scattering events on top of a highly occupied, single mode field. The highly occupied field arises from the condensate. To begin, we consider the non-interacting part of the Hamiltonian (3.19) in zero magnetic field, which is,

$$H_{NI} = \sum_{m=-1}^{1} \int d^2x \hat{\psi}_m^\dagger(x) \left[ -\frac{\hbar^2 \nabla^2}{2M} \right] \hat{\psi}_m(x),$$

and write the field operator $\hat{\psi}_m$ in terms of eigenstates of (3.33),

$$\hat{\psi}_m(x) = \sum_k \frac{1}{\sqrt{A}} e^{ik\cdot x} \hat{b}_{m,k},$$

where $\hat{b}_{m,k}$ is the destruction operator for a free particle mode in the $m$th spin state with momentum $\hbar k$, and $A$ is the system area. We assume that the ground state of the Hamiltonian (3.33) is highly occupied and so approximate it as classical by setting $\hat{b}_{m,0} \to \sqrt{N_0}\zeta_m$. Here $N_0$ is the number of particles in the ground state. The complex numbers $\zeta_m$ satisfy $\sum_m |\zeta_m|^2 = 1$ and determine how the condensate population is distributed between the spin states.

In the Bogoliubov approximation we retain scattering events that are at least second order in $\sqrt{N_0}$ in the Hamiltonian (3.19). Physically this means that we are assuming scattering events are dominated by those that involve at least two particles from the classical field. This gives a Hamiltonian that is quadratic in field operators that can be diagonalized by a Bogoliubov transformation [356]. The excited state diagonal modes of the Bogoliubov Hamiltonian are termed quasiparticle modes. These describe the low energy excitations of the Bose gas. For a spinor condensate, the different spin levels make the full Bogoliubov Hamiltonian rather cumbersome in general, and so

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we refer the reader to [252, 357, 358] for further details. For particular choices of $\zeta_m$, the result can be made simpler, and we will present these results when needed.

The Bogoliubov approximation is useful in systems with a large zero momentum mode occupation and a relatively small number of quantum excitations on top. For comparison, the classical field approximation, Sec. 3.5, describes all highly occupied modes. There can still be regimes of overlap between the two methods, when excited modes are highly populated but have a relatively low population compared to the zero momentum mode. In such cases, a perturbative expansion of the classical field equations around the zero momentum mode would be equivalent to the Bogoliubov approximation.

3.7 Ground state phases in spinor condensates

A spin-1 condensate exhibits different ground state manifolds depending on the Zeeman shifts and the spin interaction. The ground state of Eq. (3.24) will be a uniform system with constant density. Equation (3.30) imposes the constraint that the total magnetization along $z$ be conserved. The linear Zeeman shift will only affect the choice of ground state in cases where the net magnetization along $z$ is non-zero. In this thesis we will only consider cases where the net magnetization along $z$ is zero. We therefore ignore the linear Zeeman shift so that the population in the spin levels depends on the energy density term,

$$\frac{g_s}{2} |F|^2 + \sum_{m=-1}^{1} qm^2 |\psi_m|^2.$$  \hspace{1cm} (3.35)

In zero external field ($q = 0$), the ground state of a spin-1 condensate is either ferromagnetic (for $g_s < 0$) or antiferromagnetic (for $g_s > 0$). In this thesis we will be interested only in the ferromagnetic case. The ground state is then the fully isotropic ferromagnetic state with some global phase. The manifold of this ground state is SO(3) [338], see Fig. 3.2(a).\footnote{This manifold will be discussed in more detail in Chap. 5.} An external field can be used to induce a quadratic Zeeman shift, see Sec. 3.3. A small positive quadratic Zeeman shift restricts the ground state magnetization to lie in the plane transverse to the external field. The ground state manifold is then $U(1) \times U(1)$, see Fig. 3.2(b). In this phase, the length of the spin vector decreases as $q$ increases because of an increasing population in the
\(m = 0\) sublevel. A negative quadratic Zeeman shift restricts the ground state magnetization to lie along the external field, with a ground state manifold \(U(1) \times \mathbb{Z}_2\), see Fig. 3.2(c). For large positive quadratic Zeeman energy, the system is in an unmagnetized ground state with all population in the \(m = 0\) sublevel, and the ground state manifold is \(U(1)\), see Fig. 3.2(d). The unmagnetized phase is separated from the transverse magnetic phase Fig. 3.2(b) by a quantum critical point at \(q = q_0 \equiv 2|g_s|n\).

Fig. 3.3 shows the phase diagram of the four ferromagnetic phases just discussed as a function of the quadratic Zeeman energy \(q\). Below we describe in more detail the four phases present in the phase diagram.

**Polar phase**

For large positive \(q\) there is an energy cost associated with occupying the \(m = \pm 1\) states and so the ground state has all population in the \(m = 0\) state. This phase is called the “polar phase”. Any polar state can be written in the form,

\[
\psi_{\alpha} = e^{i\alpha} \sqrt{n} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},
\]

where \(\alpha \in [0, 2\pi)\). The gauge symmetry results in a \(U(1)_\alpha\) ground state manifold. Once \(q\) is reduced below \(q_0 = 2|g_s|n\), the system can lower its energy by becoming magnetized.

**Easy-plane phase**

For positive \(q\) less than \(q_0\), there is an energy cost associated with becoming magnetized along \(z\) and so the system develops a transverse magnetization \(\mathbf{F} = (F_x, F_y, 0) \equiv \mathbf{F}_\perp\). This phase is called the “easy-plane” phase. Any easy-plane state can be written in the form,

\[
\psi_{\alpha, \theta} = e^{i\alpha} e^{-i\theta f_z} \frac{\sqrt{n}}{2} \begin{pmatrix} \sqrt{1 - q/q_0} \\ \sqrt{2(1 + q/q_0)} \\ \sqrt{1 - q/q_0} \end{pmatrix},
\]

\[
= \frac{\sqrt{n}}{2} e^{i\alpha} \begin{pmatrix} e^{-i\theta} \sqrt{1 - q/q_0} \\ \sqrt{2(1 + q/q_0)} \\ e^{i\theta} \sqrt{1 - q/q_0} \end{pmatrix},
\]

(3.37)
Figure 3.2: Ground state manifolds for the four spin phases considered in this thesis. In addition to the spin degeneracy represented by the spheres, each phase is invariant under a global gauge rotation, or equivalently a spin rotation about the direction of magnetization. This is represented by the circular loop adjacent to each spin sphere. (a) With no quadratic Zeeman shift the magnetization can point anywhere on a spherical shell. Combined with the gauge degeneracy, this gives rise to an SO(3) ground state manifold. (b) With a small positive quadratic Zeeman shift, the magnetization lies in a plane perpendicular to the external field. This gives rise to a U(1) degeneracy that combines with the U(1) gauge degeneracy to form a U(1) × U(1) ground state manifold. (c) With a negative quadratic Zeeman shift, the magnetization lies parallel to the external field. This gives rise to a Z₂ degeneracy that combines with the U(1) gauge degeneracy to form a U(1) × Z₂ ground state manifold. (d) For positive quadratic Zeeman energies larger than the spin interaction energy \( q > 2|g_s|n \equiv q_0 \), the population occupies the \( m = 0 \) state and there is no spin density. There is still a gauge degeneracy and so the ground state manifold is U(1). The coordinate axes in the centre of the figure show the spin directions, assuming the external field is parallel to \( F_z \).
Figure 3.3: Ground state phase diagram of a ferromagnetic spin-1 condensate for varying quadratic Zeeman energy $q$. There is a quantum critical point at $q = q_0 \equiv 2|g_s|n$. The spheres show the ground state degeneracy in the magnetization for each phase. In addition there is a gauge degeneracy (not shown). Details of the four phases, including gauge degeneracy, are discussed in Fig. 3.2 and in the main text.

where $\alpha, \theta \in [0, 2\pi)$. The angle $\theta$ gives the direction of transverse magnetization, $F_\perp \propto (\cos \theta, \sin \theta, 0)$. The gauge angle $\alpha$ arises from either gauge rotation or equivalently a spin rotation around the direction of magnetization. The gauge and transverse spin symmetries result in a $U(1)_\alpha \times U(1)_\theta$ ground state manifold. The magnitude of the transverse spin is,

$$|F_\perp| = n \sqrt{1 - \left(\frac{q}{q_0}\right)^2}. \quad (3.38)$$

This value changes continuously from $|F_\perp| = 0$ at $q = q_0$ to the fully magnetized state $|F_\perp| = n$ as $q \to 0$.

**Easy-axis phase**

For negative $q$, the lowest energy state has full population in either of the $m = \pm 1$ states. The magnetization therefore lies along $F_z$. This phase is
called the “easy-axis” phase. Any easy-axis state can be written in the form,

$$\psi_{\alpha,s} = e^{i\alpha} e^{-is\pi f_z} \sqrt{n} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \sqrt{(1 - s)} ne^{i\alpha} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \sqrt{sn}e^{i\alpha} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad (3.39)$$

where $\alpha \in [0, 2\pi)$ and $s \in \{0, 1\}$. The gauge angle $\alpha$ arises from either gauge rotation or equivalently a spin rotation around the direction of magnetization. The integer $s$ determines the direction of easy-axis magnetization. However, the total magnetization along $z$ must be zero, and therefore for any ground state both the $s = 0$ and $s = 1$ phases must coexist in an immiscible state. The boundary between the two immisible phases costs energy and so will have some length minimizing shape. Any degeneracy in the choice of boundary should be small and affect only length scales on the order of the system size.\(^6\) Ignoring any degeneracy in this immisibility boundary, the gauge and inversion symmetries of the state (3.39) result in a $U(1) \times \mathbb{Z}_2$ ground state manifold. The easy-axis phase is fully magnetized ($|F_z| = n$).

**Isotropic phase**

At $q = 0$ the magnetization can point in any direction. This phase is called the “isotropic phase”. Any isotropic state can be written in the form,

$$\psi_{\alpha,\theta,\beta} = e^{-i\theta f_z} e^{-i\beta f_y} e^{-i\alpha f_z} \sqrt{n} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \sqrt{n} e^{i\alpha} \begin{pmatrix} e^{-i\theta \cos^2 \frac{\beta}{2}} \\ \sqrt{2} \cos^\theta \sin^\frac{\beta}{2} \\ e^{i\theta} \sin^2 \frac{\beta}{2} \end{pmatrix}, \quad (3.40)$$

where $\alpha, \theta \in [0, 2\pi)$ and $\beta \in [0, \pi)$. We could have also included an additional gauge rotation, but this is equivalent to the initial rotation by $\alpha$ about $f_z$. The manifold that results from the combination of angles $\{\alpha, \theta, \beta\}$ is $SO(3)$ \([338]\).

\(^6\)In antiferromagnetic lattices a somewhat similar degeneracy can be very important. Antiferromagnetic interactions favour adjacent spins to oppositely align, so that the presence of “immiscibility boundaries” between spins is favoured. The many ways of distributing these boundaries so as to maximise their area affects the system at short length scales and can lead to a large ground state degeneracy for certain systems, known as geometric frustration \([359, 360]\).
and will be discussed in more detail in Chap. 5. Note that unlike the easy-axis and easy-plane phases, this manifold is not a simple product of spin and gauge degrees of freedom. The isotropic phase is fully magnetized ($|F| = n$).

### 3.8 Order parameter manifolds

In the previous section we repeatedly used the term “ground state manifold” somewhat intuitively. This informal understanding will suffice for much of this thesis. However, for some of the discussions of the SO(3) manifold in Chap. 5 and 8, and in particular why this is different from an $S^2$ manifold, a formal definition is useful. We discuss here an order parameter manifold, noting that this reduces to a ground state manifold for quantum phase transitions, where the free energy is just the energy.

In general, an order parameter manifold is defined as follows [299]. We firstly consider the group $G$ of transformations that act on a particular choice of the order parameter $\psi$ that leave the free energy invariant, $F[g\psi] = F[\psi]$ for $g \in G$. We will assume that the order parameter $\psi$ is a minimum of the free energy, although this restriction can be relaxed [299, 318]. There is a subgroup $H$ (called the isotropy group) of $G$ that leaves $\psi$ unchanged, i.e. $h\psi = \psi$ for $h \in H$. The order parameter manifold $M$ is then defined as the following set,

$$M \equiv G/H = \{gH : g \in G\} = \{gh : h \in H : g \in G\}. \quad (3.41)$$

The set $M$ is a set of cosets $gH = \{gh : h \in H\}$. We assume that $M$ is smooth.

The definition (3.41) is somewhat abstract, so it is reasonable to ask why this is useful. To see this, it is worth considering why the group $G$ is not sufficient to define an order parameter manifold. The reason is that $G$ contains superfluous elements that leave $\psi$ invariant, which is the isotropy group. We wish to omit these elements (apart from the identity) from the order parameter manifold. There is a one to one correspondence between the elements we wish to retain and the distinct cosets $gH$. It is this reason that the definition (3.41) is reasonable.

It is not necessary for the elements of $M$ to form a group under some binary operation. In fact, it is only possible to do this if the left and right cosets $gH$ and $Hg \equiv \{hg : h \in H\}$ are equal. The binary operation in this case is $(aH)(bH) = a(Hb)H = a(bH)H = (ab)HH = (ab)H$ for $a, b \in G$ [361]. In such cases, the elements of $M$ form a Lie group. For any $h \in H$, the coset $hH$ corresponds to the identity element of the group.
Treating the order parameter manifold as a set of cosets is often not the most convenient treatment. We will usually work with some smooth manifold $\mathcal{M}$ that is diffeomorphic to $M$. Two smooth manifolds are diffeomorphic if there is a smooth isomorphism between them, which in such cases is called a diffeomorphism. The transformations $s \in G$ that map onto the distinct cosets $gH$ in $M$ form such a manifold. Alternatively, the set of possible order parameters $s\psi$ also forms a manifold.

The order parameter manifold of a classical Heisenberg model in the ferromagnetic state displays many of the features just discussed. The Hamiltonian of the classical Heisenberg model is invariant under O(3) transformations. Only a subset of these correspond to the order parameter manifold, namely rotations that are not around the direction of magnetization. The elements of this manifold do not form a group. The order parameter manifold can also be represented by the spherical surface $S^2$ if we consider the space that the order parameter maps out. In comparison, the isotropic state (3.40) gives rise to an SO(3) manifold. The difference between this state and the classical Heisenberg model arises from the nontrivial effect on (3.40) of a spin rotation around the direction of magnetization. Such rotations are therefore included in the order parameter manifold, whereas they are not in the manifold of the classical Heisenberg model. As we will discuss in Chap. 5, this difference has profound effects on the properties of the spin-1 isotropic phase, in particular the topological defects that are supported.

As a final point, it is possible that the set of transformations in the manifold is best understood as being made up of ordered pairs of transformations $(p, q)$, where $p$ and $q$ reflect two distinct symmetries. In such cases, we can write the order parameter manifold as a Cartesian product of simpler manifolds $P$ and $Q$, i.e. $(p, q) \in P \times Q = \mathcal{M}$. For example, the easy-axis and easy-plane ground state manifolds introduced in the previous section take the form $U(1) \times Q$ where $U(1)$ arises from gauge symmetry and $Q$ arises from spin symmetry ($Q = \mathbb{Z}_2$ for the easy-axis phase and $Q = U(1)$ for the easy-plane phase).

### 3.9 Spin currents and vortices

#### 3.9.1 Hydrodynamics and vortices in scalar condensates

In the GPE formulation, a BEC is a fluid and, like a conventional fluid, may be described by hydrodynamic equations [356].\(^7\) Interpreting the fluid flow of...
a BEC in terms of a conventional fluid, while still appreciating the important
differences between the two systems, leads to a powerful understanding of
BEC dynamics. Considering first a scalar BEC, there are three features that
distinguish a scalar BEC from a conventional fluid.

**Quantization of vorticity.** The fluid velocity in a BEC is proportional to
the gradient of the condensate phase \( \alpha \) (for a scalar BEC there is only this
single phase),

\[
v = \frac{\hbar}{M} \nabla \alpha. \tag{3.42}
\]

As a result the vorticity in the system is quantized, since single valuedness
of the condensate wavefunction fixes \( \oint_C \mathbf{dl} \cdot \nabla \alpha \) to be an integer multiple of
\( 2\pi \), where \( C \) is any closed loop. The quantization of vorticity reflects the
quantization of angular momentum in the condensate. The vorticity in a
scalar condensate is,

\[
\omega \equiv \nabla \times \mathbf{v} = \sum_i \frac{2\pi \hbar \kappa_i}{M} \delta(\mathbf{r} - \mathbf{r}_i), \tag{3.43}
\]

where \( \mathbf{r}_i \) denote points around which the fluid velocity circulates by \( 2\pi \kappa_i \),
\( \kappa_i \in \mathbb{Z} \). We can therefore picture the vorticity as arising from quantized
vortices at positions \( \mathbf{r}_i \). Note that the condensate density at the centre of a
vortex must be zero to avoid a kinetic energy divergence associated with the
vorticity divergence. Although we label a vortex by its core position \( \mathbf{r}_i \), a
vortex is really an extended object defined by the circulation of the fluid.

In a two dimensional scalar condensate, a vortex is a topological defect
(as discussed in Sec. 2.6.3) and so cannot be removed by local changes to
the system. The vanishing density at the vortex core reflects that the vortex
is a defect in the order parameter manifold, where for this discussion the
order parameter manifold is understood as the condensate wavefunction. The
charge of the defect is \( \kappa_i \).

**No viscosity.** The GPE formulation of a BEC describes conservative evolu-
tion of the fluid. A lack of dissipation means there is no viscosity in the
hydrodynamics of a BEC. This allows for superfluid flow of the condensate.

conservation laws. The continuous fields arise naturally in the GPE formulation of a BEC,
which is described by a density field and a phase field. In comparison, a conventional fluid
can be described by continuous fields in regimes where local equilibrium is established on
much faster time scales than the flow of conserved quantities [356].
The lack of viscosity also plays an interesting role in the study of quantum turbulence in BECs [362–366].

**Modified pressure term.** Like a conventional dissipationless fluid, the fluid flow in a BEC may be described by an Euler equation, with changes in fluid flow driven by pressure gradients. In a BEC there are two contributions to the pressure term that appears in the Euler equation. Firstly, the particles of a BEC undergo two body interactions. A change in volume of a BEC, keeping particle number fixed, leads to a change in the energy of the two body interactions. Therefore the two body interactions manifest themselves as a pressure. A second contribution to the pressure arises from the kinetic energy of a BEC. The kinetic energy of a BEC is made up of a contribution from the superfluid velocity and a contribution from spatial variation of the density profile of the condensate. This second contribution is not present in a conventional fluid and affects the superfluid velocity in the same way as a pressure so is termed the *quantum pressure* [356].

### 3.9.2 Hydrodynamics and vortices in spinor condensates

In a spinor condensate, the gradient of the phase of each spin component gives rise to a superfluid current of that component. Superpositions of these currents can then give rise to a flow of different physical observables. For a spin-1 condensate, there are three spin states giving rise to three currents. Like in the scalar case, a hydrodynamic formulation of a spin-1 condensate can be developed using currents and densities of mass, spin and nematic quantities [367]. In this thesis we will be interested in mass and spin densities and currents. The mass current is given by,

\[
\mathbf{v}_M = \frac{\hbar}{2Mn} \sum_{m=-1}^{1} \left( \psi_m^* \nabla \psi_m - \psi_m \nabla \psi_m^* \right)
\]

\[
= \frac{\hbar}{4\pi} \sum_{m=-1}^{1} \text{Im} \left( \psi_m^* \nabla \psi_m \right).
\] (3.44)

The condensate mass is conserved and so the mass current obeys a continuity equation with no source or sink,

\[
\frac{\partial n}{\partial t} + \nabla \cdot (n \mathbf{v}_M) = 0.
\] (3.45)
The three types of spin currents are given by,

\[ \nu_f = \frac{\hbar}{2M n_i} \sum_{m,m'=-1}^{1} \left( \psi_m^* (f_\mu)_{mm'} (\nabla \psi_{m'}) - (\nabla \psi_m^*) (f_\mu)_{mm'} \psi_{m'} \right) \]

\[ = \frac{\hbar}{M n} \sum_{m,m'=-1}^{1} \text{Im} \left( \psi_m^* (f_\mu)_{mm'} \nabla \psi_{m'} \right). \]  

(3.46)

Each spin current obeys a continuity equation,

\[ \frac{\partial F_\mu}{\partial t} + \nabla \cdot (n \nu_f) = -\frac{2q}{\hbar} \epsilon_{z\mu \nu} N_{z\nu}, \]  

(3.47)

where,

\[ N_{z\nu} = \sum_{m,m'=-1}^{1} \psi_m^* (f_\nu f_\mu)_{mm'} \psi_{m'}. \]  

(3.48)

For non-zero \( q \), the source term in Eq. (3.47) reflects that \( F_x \) and \( F_y \) need not be conserved. The source term vanishes for \( \mu = z \), consistent with the conservation of \( F_z \). Spin-1 condensates give rise to a rich variety of vortices arising from the different possible fluid flows. Such vortices will play an essential role in the phase ordering dynamics of spin-1 condensates. We will discuss each relevant type of vortex in the results chapters.

3.10 Phase ordering dynamics in ferromagnetic spin-1 condensates: early time dynamics

In the previous sections we have presented the foundation for a classical field study of phase ordering dynamics in ferromagnetic spin-1 condensates. In this section we will review the previous research into the early time phase ordering dynamics, which has paved the way for the results in this thesis.

Early experimental work in the phase ordering dynamics in spin-1 ferromagnetic condensates was conducted by the Stamper-Kurn group in UC Berkeley [223, 283, 295, 368, 369]. In their initial work [223], the Berkeley group prepared an \(^{87}\text{Rb} \) condensate in the \( m = 0 \) (unmagnetized) state in a strong quadratic Zeeman field, \( q \gg q_0 \). The quadratic Zeeman energy was then rapidly quenched to a value \( 0 < q \ll q_0 \) and the spontaneous formation of easy-plane magnetic domains was observed, see Fig. 1.2(b). Consistent with the development of domains, the group observed vortices consisting
of a circulation of the transverse magnetization direction, Fig. 3.4(a). The transverse magnetization density was found to grow exponentially as unstable modes were populated, Fig. 3.4(b). This growth saturated after a time on the order of the spin interaction energy.

In a later experiment by the Berkeley group, phase ordering dynamics in the easy-axis, isotropic and easy-plane phase was explored [295]. In this experiment the initial instability was generated from a temperature quench, rather than the quadratic Zeeman energy quench used in [223]. Figure 1.3(a) shows the evolution of transverse and longitudinal spin domains in each of the three phases. For later evolution times the ground state magnetization in each phase dominates. The early stages of domain growth at zero quadratic Zeeman energy was also measured in this work, Fig. 1.3(b). However, the duration of the experiment was too short to explore the long time phase ordering dynamics.

Classical field methods have been used to numerically simulate a quench from the polar to the easy-plane phase. Predictions of the growth of magnetization and average domain size from these simulations were found to be in reasonable agreement with the experimental results from the Stamper-Kurn group [226, 285].

A simple and quantitative theoretical understanding of the quench results can also be developed using a Bogoliubov analysis (see Sec. 3.6) [224, 225, 227, 284]. This analysis is applicable to the early time dynamics following quenches to the easy-plane, isotropic and easy-axis phases. Initially, the spin-1 system is in the $m = 0$ state, and so we perform a Bogoliubov expansion about this state in the zero momentum mode. The dynamics of modes in the $m = \pm 1$ states are then governed by the Bogoliubov Hamiltonian [224],

$$H = \sum_k \left( \epsilon_k - pm + q + gn_0 \right) \left( \hat{b}_{1,k}^\dagger \hat{b}_{1,k} + \hat{b}_{-1,k}^\dagger \hat{b}_{-1,k} \right) + g_x n_0 \sum_k \left( \hat{b}_{1,k}^\dagger \hat{b}_{-1,-k}^\dagger + \hat{b}_{1,k} \hat{b}_{-1,-k} \right).$$

Here $\epsilon_k \equiv \hbar^2 k^2 / 2M$ is the free particle spectrum, $n_0$ is the mean condensate density and $\hat{b}_{m,k}$ is a destruction operator for a free particle mode in the $m$th spin state with momentum $\hbar k$. This takes the same form as the Hamiltonian for a non-degenerate parametric amplifier from quantum optics [370], which describes the amplification of modes from some coherent source. In quantum optics the coherent source would be a laser; in our case it is the condensate in the polar state.

The Hamiltonian (3.49) is quadratic in annihilation and destruction operators. The Heisenberg equations of motion for $\hat{b}_{\pm 1,k}$ are therefore linear and
Figure 3.4: Experimental observation of vortices and growth of local magnetization density following a quench from the polar to the easy-plane phase. (a) Left panel: phase gradient of transverse spin across the condensate at a time of 156 ms after the quench, showing a vortex of transverse spin within the yellow box (region magnified in the right panel). (b) Growth of mean square transverse (filled rectangles) and longitudinal (unfilled circles) magnetization density. The solid grey curve is an exponential fit to the initial growth of transverse magnetization. Figure adapted from [223].
can be solved to obtain [227],
\[
\hat{b}_{\pm 1,k}(t) = \left( \cos \frac{E_k t}{\hbar} - i \frac{\epsilon_k + q + g_s n_0}{E_k} \sin \frac{E_k t}{\hbar} \right) \hat{b}_{\pm 1,k}(0) \\
- \left( i \frac{g_s n_0}{E_k} \sin \frac{E_k t}{\hbar} \right) \hat{b}^\dagger_{\mp 1,-k}(0),
\] (3.50)
where \(E_k = \sqrt{(\epsilon_k + q)(\epsilon_k + q + 2g_s n_0)}\). The population is initially entirely in the polar state, and so the initial state satisfies
\[
\langle \hat{b}^\dagger_{\pm 1,k}(0) \hat{b}_{\pm 1,k}(0) \rangle = 0,
\]
\[
\langle \hat{b}_{\pm 1,k}(0) \hat{b}^\dagger_{\pm 1,k}(0) \rangle = 1.
\] (3.51)

At time \(t = 0\) the quadratic Zeeman energy is instantaneously quenched to \(q < q_0\). Modes with \(k < k_c \equiv \sqrt{2M(q_0 - q)/\hbar}\) then have imaginary \(E_k\) and are unstable. The population in these modes will grow exponentially,
\[
\langle \hat{\psi}^\dagger_{\pm 1}(x,t) \hat{\psi}_{\pm 1}(x,t) \rangle = \frac{1}{A} \sum_k \left| \frac{g_s n_0}{E_k} \sin \frac{E_k t}{\hbar} \right|^2 e^{2iE_k t/\hbar} + \text{stable modes},
\] (3.52)
where \(A\) is the system area. For times \(|E_k t/\hbar| \gg 1\) the stable modes can be ignored. The unstable modes grow with a time constant \(\tau \sim \hbar/q_0 \equiv t_s\).

Atoms in the \(m = 0\) condensate mode are interacting and scattering into the \(m = \pm 1\) states, resulting in exponential growth of the population in the \(m = \pm 1\) states. As a consequence of this, a transverse magnetization develops. We measure the growth of transverse magnetic order using the transverse spin spatial correlation function,
\[
G_\perp(x,x',t) \equiv \frac{1}{n_0^2} \left\langle \hat{F}_\perp(x,t) \cdot \hat{F}_\perp(x',t) \right\rangle \\
= \frac{2}{n_0 A} \sum_k \left| \cos \frac{E_k t}{\hbar} + i \frac{\epsilon_k + q}{E_k} \sin \frac{E_k t}{\hbar} \right|^2 e^{i k (x - x')} \\
\approx \frac{1}{2n_0 A} \sum_{k < k_c} \frac{q_0}{q_0 - q - \epsilon_k} e^{2iE_k t/\hbar + i k (x - x')},
\] (3.53)
where in the third line we have neglected stable modes. Correlations are translationally and rotationally invariant, so depend only on the magnitude \(r \equiv |x - x'|\). For a sufficiently large system the wavevectors are closely spaced
and we can write the summation in Eq. (3.53) as an integral, including the phase space volume $4\pi^2/A$. We can then use the identity,

$$\int d\theta e^{iu\cos\theta} = 2\pi J_0(u), \quad (3.54)$$

where $J_0$ is the zeroth order Bessel function of the first kind, to obtain,

$$G_\perp(r,t) = \frac{1}{4\pi n_0} \int_0^{k_c} dk \frac{q_0 k}{q_0 - q - \epsilon_k} J_0(kr) e^{2|E_k|t/\hbar}. \quad (3.55)$$

Equation (3.55) describes the initial formation of transverse magnetic domains.

For $q < q_0/2$, the fastest growing mode (i.e. the mode that maximises $|E_k|$) has a wavevector,

$$k_f = \sqrt{\frac{2M}{\hbar^2}} \left(\frac{q_0}{2} - q\right). \quad (3.56)$$

An approximation to the integral in Eq. (3.55) can be obtained using the saddle point approximation about this fastest growing mode [227]. This gives a spatial dependence $J_0(kfr)$ so that the transverse spin is initially correlated over a distance $\xi \sim \hbar/\sqrt{Mq_0} \equiv \xi_s$. For $q_0/2 \leq q < q_0$, the fastest growing mode is $k_f = 0$. Correlations then spread out diffusively ("light-cone spreading") with a diffusion coefficient on the order of $\xi_s^2/t_s$ [227]. For times $t \sim t_s$ where the Bogoliubov theory is valid, this also leads to correlations over a distance $\xi \sim \xi_s$.

In addition to the formation of transverse spin correlations across a length scale $\xi \sim \xi_s$, there is an exponential growth of magnetization with a time constant $\tau \sim t_s$. Evaluating Eq. (3.55) at $r = 0$ gives the mean square transverse magnetization density as a function of time. Once the population in the $m = \pm 1$ levels becomes comparable to that in the $m = 0$ level, the Bogoliubov results are no longer valid.

As the population in the $m = \pm 1$ states grows, there is also growth in the magnetization along $z$. Correlations of $F_z$ density could be computed by Eq. (3.50), but are cumbersome. Here we simply estimate the local growth of $F_z$ density. The magnetization grows as,

$$\langle \hat{F}_z(x,t) \hat{F}_z(x,t) \rangle \sim \langle \hat{a}^\dagger_{\pm 1,k} \hat{a}_{\pm 1,k} \hat{a}^\dagger_{\pm 1,k} \hat{a}_{\pm 1,k} \rangle \sim \frac{1}{n_0 \xi_s^2} e^{t/t_s}. \quad (3.57)$$

Therefore the growth of $F_z$ mean square magnetization is also exponential, but with a time constant $2t_s$ and a density suppressed by $1/n_0 \xi_s^2$ relative to the transverse spin density.
Barnett et al. [296] used the classical field method to simulate spin dynamics following a quench of the quadratic Zeeman energy from the polar phase to the easy-plane phase in a homogeneous system. In this work they found good agreement between classical field simulations and the Bogoliubov prediction of exponential growth of magnetization densities, during the early time dynamics.

We have also carried out such simulations, by numerically solving the spin-1 GPEs, Eq. (3.30). Quantum fluctuations (Eq. (3.51)) seed the growth of domains, and can be modelled by adding noise to the initial state, see Sec. 3.11 for details. Figure 3.5(a) shows the evolution of transverse spin direction following a quench from deep in the polar phase to the easy-plane phase with \( q = 0.3q_0 \). Small domains of transverse magnetization quickly form, with spatial correlations over a length scale \( \xi \sim \xi_s \), frames (i)–(iii). For longer times, the transverse spin domains begin to grow, frame (iv). The spatial correlation of transverse spin domains is shown in Fig. 3.5(b) at time \( t = 9t_s \). Clearly visible is a characteristic Bessel function shape, and the result is described well by the Bogoliubov approximation (3.55). For shorter times there is good agreement except at \( r = 0 \), due to a contribution to correlations from stable modes that are not accounted for in Eq. (3.55). For times \( t \gtrsim 12t_s \) the population in the \( m = \pm 1 \) modes becomes appreciable and the Bogoliubov approximation becomes invalid. Associated with the formation of transverse spin domains is a growth of local mean square spin density \( \langle F^2_{\perp} \rangle \), Fig. 3.5(c). The initial exponential growth agrees well with the Bogoliubov result (3.55). There is also growth of \( F_z \) spin density, although the magnitude is much smaller than the transverse spin density. For times \( t \gtrsim 12t_s \) the exponential growth of spin density saturates and the Bogoliubov approximation becomes invalid.

### 3.11 General details of numerics

The work we present in this thesis is heavily based on results from computer simulations of the spin-1 GPEs (3.30). In Chaps. 4, 5 and 7 the spin-1 GPEs are integrated using an adaptive step Runge-Kutta method [371]. For each time step this method evaluates a Runge-Kutta integrated solution to both fourth and fifth order in the time step, and computes the error in the fourth order calculation over the fifth order calculation. The time step is then adapted until the error drops below some tolerance. We take the error to be the maximum relative error in the wavefunction amplitude across our numerical grid (for wavefunction amplitudes not close to zero) and use a tolerance of \( 10^{-6} \). This results in time steps of approximately \( 10^{-1}t_n \), where
Figure 3.5: Early time dynamics following a quench from the polar phase to the easy-plane ferromagnetic phase, obtained from numerical simulation of the spin-1 GPEs. (a) Evolution of transverse spin order. Colour represents spin direction according to the colour wheel adjacent to frame (iv). The initial noise in (i) seeds the formation of local order, and correlations of size $\xi \sim \xi_s$ quickly develop after the quench, (ii) and (iii). For later times, the correlation length has grown, (iv). (b) Spatial correlations of transverse spin at time $t = 9\xi_s$ (blue dots), showing good agreement with the Bogoliubov result (black line). (c) Growth of local mean square magnetization densities $\langle F^2_\perp \rangle$ (red line) and $\langle F^2_z \rangle$ (blue line). The initial growth of $F_\perp$ agrees well with the Bogoliubov prediction (3.55) (black dashed line). The $F_z$ magnetization also grows although its density remains much lower than the density of $F_\perp$. For later times the growth of magnetization density saturates.
$t_n = 1/g_n n$ is the density interaction time scale (note $t_n \ll t_s$ where $t_s$ is the spin interaction time scale).

The microcanonical results in Chap. 6 require very long simulation times. We therefore use a fourth order symplectic integrator that is more effective at conserving constants of motion for long simulation times [372]. For this fixed step algorithm we use a time step of $0.1t_n$. The grand canonical results in Chap. 6 are integrated using the adaptive step Runge-Kutta method described above.

We evaluate all simulations on a square numerical grid with equally spaced grid points. We use periodic boundary conditions and evaluate the spatial derivatives in the kinetic energy operator with spectral accuracy using fast Fourier transforms. The simulations were predominantly done on Nvidia graphics cards (Tesla K40c or GeForce GTX Titan Black) using MATLAB’s GPU capability. A single trajectory simulation took between roughly one day and one week, depending on the simulation.

In Chaps. 4, 5 and 6 we simulate quenches from the polar phase to one of three ferromagnetic phases. For our initial state, we take

$$\psi(x) = \sqrt{n_0} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \delta(x), \quad (3.58)$$

where $\sqrt{n_0} (0, 1, 0)^T$ is the polar condensate wavefunction and $\delta$ is a small noise field given by

$$\delta(x) = \sum_k \begin{pmatrix} \alpha_k^+ e^{i k \cdot x} \\ \alpha_k^0 u_k e^{i k \cdot x} - \alpha_k^0 v_k e^{-i k \cdot x} \\ \alpha_k^- e^{i k \cdot x} \end{pmatrix}. \quad (3.59)$$

Here the $\{\alpha_k^m\}$ are independent complex Gaussian random variables with

$$\langle \alpha_{k'}^m \alpha_k^m \rangle = \frac{1}{2} \delta_{mm'} \delta_{k'k}. \quad (3.60)$$

The amplitudes $\{u_k, v_k\}$ are given by

$$u_k = \sqrt{\frac{\epsilon_k + g_n n_0}{2 \sqrt{\epsilon_k (\epsilon_k + 2g_n n_0)}} + \frac{1}{2}}, \quad v_k = \sqrt{u_k^2 - 1}, \quad (3.61)$$

with $\epsilon_k = \hbar^2 k^2 / 2M$. For $k = 0$ we take $u_k = v_k = 0$. The noise added this way corresponds to adding a half-quantum of occupation to the Bogoliubov
modes for the polar phase at large $q_i$ [224, 356] as per the truncated Wigner prescription [328]. This mimics quantum fluctuations that seed noise in the Bogoliubov theory, resulting in the formation of symmetry breaking domains. Most of the results in Chaps. 4, 5 and 6 are calculated by averaging over solution trajectories of the spin-1 GPEs calculated with independent sampling of the initial noise. The number of trajectories varies from four to ten and is stated in the chapters.
Chapter 4

Coarsening dynamics in the easy-axis and easy-plane phases

In this chapter we examine the dynamics of a quasi-two-dimensional spin-1 condensate in which the quadratic Zeeman energy \( q \) is suddenly quenched to either an easy-axis or easy-plane ferromagnetic phase. We identify scale invariant growth of domains in both ferromagnetic phases during the late stages of phase ordering. We study the quench dynamics for a variety of \( q \) values and show that there is a single dynamic critical exponent to characterize the scale invariant domain growth for each phase. For both quenches we give simple analytic models that capture the essential scale invariant dynamics, and correctly predict the exponents. Because the order parameter for each phase is different, the nature of the domains and the relevant topological defects in each type of coarsening is also different. To explore these differences we characterize the fractal dimension of the domain walls, and the relationship of polar-core spin vortices to the domains in the easy-plane phase. Finally, we consider how the energy liberated from the quench thermalizes in the easy-axis quench. We show that local equilibrium is established in the spin waves on moderate time scales, but continues to evolve as the domains anneal.

4.1 Introduction

After a rapid quench through a symmetry-breaking phase transition a many-body system will form causally disconnected spatial domains, each making an independent choice for the symmetry broken order parameter. The coarsening dynamics of how such a system subsequently evolves towards equilibrium is an area of broad interest [37]. At long times after the quench a universal scaling regime can develop: correlation functions of the order parameter
collapse onto a universal scaling function (independent of time $t$) when space is scaled by a characteristic length $L(t)$. The growth law for this characteristic length $L(t) \sim t^{1/z}$ yields the dynamic critical exponent $z$ (see Sec. 2.6 for a more thorough discussion). While most theories for coarsening dynamics have been developed for dissipative models related to temperature quenches, recently there has been growing interest in the dynamics of systems under conservative Hamiltonian evolution, particularly due to developments with ultra-cold atomic gases [187, 214, 217, 373–375].

In this chapter we explore the coarsening dynamics of a ferromagnetic spin-1 condensate in both the easy-axis and easy-plane phases. We demonstrate that both quenches behave universally in their late time coarsening dynamics. We find $1/z = 2/3$ for the easy-axis phase, consistent with a binary fluid interpretation [376] and disagreeing with an earlier result of $1/z \approx 1/3$ [228]. A hydrodynamic analysis [231] also obtained a $t^{2/3}$ growth law, and showed that the growth reduces to $t^{1/3}$ if the effects of superfluid flow are removed. A recent study of the coarsening dynamics of an immiscible binary condensate revealed a $t^{2/3}$ growth law and verified the scaling hypothesis by demonstrating correlation function collapse [217]. For the easy-plane case, we show that topological defects, namely polar-core spin vortices, play a crucial role in the dynamics, and find an exponent (accounting for logarithmic corrections to scaling) of $z = 1$, consistent with the model E dynamic universality class [19]. We simulate quenches for a wide range of $q$ values in the easy-axis and easy-plane regimes, and demonstrate that the exponents obtained are universal. Supporting these results we develop models of the key processes governing coarsening and apply scaling arguments to obtain the same exponents.

We consider the mean value and fluctuations of the magnetization in the post-quench system to reveal the thermalization of the energy liberated by the quench. For the easy-axis case we develop a scheme for thermometry using spin-waves and demonstrate that these modes thermalize on a much faster time scale than the order parameter evolution governed by the coarsening dynamics. We also consider the domain wall structure by evaluating the order parameter structure factor, where the domain wall properties are revealed by a Porod tail feature. Interestingly, the analysis of the Porod tail for the easy-plane case suggests that the domain walls have a fractal structure. We verify this by directly applying a box-counting algorithm to the spatial domains of the coarsened system. The easy-plane phase has an order parameter that supports vortices (polar-core vortices) as topological defects. We evaluate the number of topological defects during the coarsening evolution and show that this is directly related to the coarsening length scale of domains.

This work we report here provides a thorough analysis of the coarsening dynamics for the ferromagnetic spin-1 condensate in the easy-axis and easy-
plane phases, and establishes a firm basis and set of tools for future work on other quenches (e.g. temperature quenches) and for work on the antiferromagnetic and higher spin cases.

The outline of this chapter is as follows. In Sec. 4.2 we introduce the relevant order parameters and their symmetries for the two phase transitions we explore, and provide details of the numerical methods we use to simulate the phase transition dynamics. The main results are presented in Sec. 4.3. We begin by examining the growth of local order following a quench in the quadratic Zeeman energy. We then introduce the order parameter correlation functions and examine the nature of dynamic scaling in the post-quench coarsening dynamics. We examine the role of vortices in the easy-plane quench and also examine the fractal dimension of domain boundaries in quenches via the order parameter structure factor and by applying a box-counting algorithm directly to the domains. For both phase transitions we develop analytic models for the relevant degrees of freedom during the coarsening regime. Dimensional analysis of these models yields the dynamic critical exponents found in the simulations. Finally, we examine the thermalisation that occurs in the easy-axis quench. We find that spin-waves in the easy-axis system thermalise on a rapid timescale compared to the order parameter coarsening dynamics. We summarise our results in Sec. 4.4 and discuss the outlook for future work in this area.

4.2 Formalism

4.2.1 Phase diagram, symmetries and conservation laws

The system we consider is a homogeneous quasi-two-dimensional spin-1 condensate described by the Hamiltonian (see Secs. 3.1-3.4),

$$H = \int d^2x \left[ \psi^\dagger \left( -\frac{\hbar^2 \nabla^2}{2M} + qf_z^2 \right) \psi + \frac{g_n}{2} n^2 + \frac{g_s}{2} |F|^2 \right],$$

(4.1)

where $\psi = (\psi_1, \psi_0, \psi_{-1})^t$. In the classical field approximation the dynamics are governed by the spin-1 GPEs (see Sec. 3.5),

$$i\hbar \frac{\partial \psi}{\partial t} = \left( -\frac{\hbar^2 \nabla^2}{2M} + qf_z^2 + g_n n + g_s F \cdot f \right) \psi.$$  

(4.2)

The Hamiltonian (4.1) gives rise to different magnetic ground states depending on the value of $q$. These ground states are discussed in Sec. 3.7. In this chapter we will explore the ordering properties of the easy-axis and easy-plane phases, see Fig. 4.1. As discussed in Sec. 3.7, degeneracy in ground state
Figure 4.1: Phase diagram and ground states of a spin-1 condensate relevant for this chapter. The spheres show the direction of magnetization in the three states. (a) For \( q < 0 \) the magnetization lies along the \( F_z \) axis and the state is termed *easy-axis*. (b) For \( 0 < q < q_0 \) the magnetization lies in the transverse \((F_x-F_y)\) plane and the state is termed *easy-plane*. (c) For \( q > q_0 \) the \( m = \pm 1 \) levels are unoccupied and the system is unmagnetized. This state is termed *polar*.
spin gives rise to a $\mathbb{Z}_2$ manifold in the easy-axis phase and a U(1) manifold in the easy-plane phase. Both phases have an additional U(1) degeneracy arising from gauge symmetry. In this chapter we will focus solely on the spin order and therefore will not analyse this additional gauge symmetry. Recent work by another student in our group has explored phase ordering of gauge symmetry in the easy-axis phase [377].

Evolution under Eq. (4.2) preserves $\mathbb{Z}_2$ symmetry corresponding to reflections in the transverse plane and U(1) symmetry corresponding to rotations about the $z$ axis. Therefore an initial state adhering to these symmetries will maintain these symmetries. For the ferromagnetic states (a) and (b), a symmetry in the system is broken when the system chooses a ground state within the ground state manifold. The nature of the ground state manifold determines this symmetry. In the easy-axis phase the order parameter is $\propto F_z$ and a choice of ground state breaks the $\mathbb{Z}_2$ symmetry. In the easy-plane phase the order parameter is $\propto (F_x, F_y)$ and a choice of ground state breaks the U(1) symmetry. The order parameter in the easy-axis phase is conserved under the evolution of (4.2), whereas the order parameter in the easy-plane phase is not. The symmetry and conservation properties of the order parameter determine the critical behaviour of the transition to each phase.

### 4.2.2 Details of simulation method

In this chapter we consider the phase ordering dynamics after a quench of the quadratic Zeeman energy from an initial value $q_i > q_0$ where the ground state is polar to a final value $q < q_0$, as shown schematically in Fig. 4.1. The system will then order into either the easy-axis phase (for $q < 0$) or the easy-plane phase (for $0 < q < q_0$). We simulate the dynamics using the GPEs (4.2) with noise added to the initial state to model quantum fluctuations and seed the growth of symmetry breaking domains, see Sec. 3.11 for details.

In experiments with $^{87}$Rb the spin interaction is much weaker than the density interaction with $g_n/|g_s| \sim 100$ [335]. To observe universal dynamics we must simulate our system over many spin times $t_s \equiv \hbar/2|g_s|n_0$. With large density interaction the system has to resolve fast but largely unimportant density fluctuations. This slows down the numerics substantially. To allow faster simulations, we use the more moderate ratio of interaction parameters $g_n/|g_s| = 10$. We have also run simulations with $g_n/|g_s| = 3$ and obtained consistent results [300]. Density fluctuations may add noise to order parameter correlations for a single simulation, but results obtained by averaging over simulations should remove this. We expect little change in the phase ordering dynamics for higher interaction parameter ratios, which would reduce the density fluctuations leading to less noise in single simulations.
We use a condensate density of \( n_0 = 10^4/\xi_s^2 \), where \( \xi_s \equiv \hbar/\sqrt{2|g_s|n_0M} \) is the spin healing length. To numerically evolve the GPEs we represent each component of the spinor field \( \psi \) on a 2D square grid with dimensions \( l \times l \) covered by an \( N \times N \) grid of equally spaced points. For simulations of quenches to the easy-axis phase we use grids of size \( l = 800\xi_s \) with \( N = 1024 \) points. For the easy-plane cases we use \( l = 1600\xi_s \) with \( N = 2048 \) points. The quadratic Zeeman energy is set at the final quench value \( q < q_0 \) for the duration of the simulation dynamics, so that the quench is effectively instantaneous at \( t = 0 \). Further details of the numerics are given in Sec. 3.11.

### 4.3 Results and Analysis

#### 4.3.1 Post-quench growth of magnetization

Following the quench to either the easy-axis or easy-plane phase, the system develops local magnetization (i.e. the spin density becomes non-zero). The development of the magnitude of the transverse \( F_\perp = (F_x, F_y) \) and longitudinal \( F_z \) magnetization is shown in Fig. 4.2 (see Sec. 3.10 for a thorough discussion of these early time dynamics). The initial growth is exponential \([227, 296]\) (also see \([224–226, 229]\)) and is similar for quenches to values of \( q \) in the easy-axis and easy-plane regimes [see Figs. 4.2(a),(b)]. The exponential growth ceases after a time \( t \approx 14t_s \). On longer time scales the magnetization develops an easy-axis [i.e. \( F_z \) dominates, Fig. 4.2(c)] or easy-plane [i.e. \( F_\perp \) dominates, Fig. 4.2(d)] character, revealing the preferred order of the phase the system has been quenched into. The magnetization reaches a steady magnitude after a time \( t \sim 200t_s \).

Immediately after the quench, the system is still in the polar phase and so is out of equilibrium. This gives the system an energy in excess of that of the ground state. The excess energy can be calculated from (4.1) yielding

\[
\Delta E = \begin{cases} 
\left( \frac{1}{4}q_0 - q \right) n_0 l^2, & q < 0, \\
\frac{1}{4}q_0 \left( 1 - q/q_0 \right)^2 n_0 l^2, & 0 < q < q_0.
\end{cases}
\]

This excess energy is available for thermalization and heats the system. This results in fluctuations of the local magnetization [Fig. 4.3(a),(b)] and a reduction in the magnitude of the magnetization from the ground state value [Fig. 4.3(c),(d)]. Both effects are more pronounced for deeper quenches (i.e. to lower \( q \) values) which have higher values of \( \Delta E \).
Figure 4.2: Growth of magnetization following a quench of a spin-1 condensate. Plain lines correspond to $F_z$ magnetization; lines with dots correspond to $F_\perp$ magnetization. Short-time dynamics for a quench to the (a) easy-axis phase ($q = 0.3q_0$) and (b) easy-plane phase ($q = -0.3q_0$). Dashed lines are guides to the eye for the initial exponential growth, with e-folding times of 1.11$t_s$ and 0.62$t_s$ respectively. Long-time dynamics for the (c) easy-axis phase and the (d) easy-plane phase. In all cases the (local) magnetization is calculated as a spatial average over the system at each time, i.e. \( \langle F^2 \rangle = \int d^2r F^2(\mathbf{r}) \) for $\nu = z, \perp$. In (c), red data is for $q = -0.3q_0$, green data is for $q = -0.6q_0$, light blue data is for $q = -1.2q_0$ and purple data is for $q = -1.2q_0$. In (d), red data is for $q = 0.1q_0$, green data is for $q = 0.3q_0$, light blue data is for $q = 0.6q_0$ and purple data is for $q = 0.9q_0$. 

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Figure 4.3: Normalized distribution of local magnetization long after the quench. The distribution of the (a) local longitudinal magnetization for easy-axis quenches and (b) local transverse magnetization for easy-plane quenches. These results are evaluated by making a histogram of the magnetization magnitudes sampled over space. In (a), red (dot-dashed) data is for $q = -0.3q_0$, green (dotted) data is for $q = -0.6q_0$, light blue (dashed) data is for $q = -1.2q_0$ and purple (solid) data is for $q = -1.2q_0$. In (b), red (dot-dashed) data is for $q = 0.1q_0$, green (dotted) data is for $q = 0.3q_0$, light blue (dashed) data is for $q = 0.6q_0$ and purple (solid) data is for $q = 0.9q_0$. In (c) and (d) the mean value and spread (standard deviation) obtained from the results in (a) and (b) are compared to the ground state magnetization (solid black lines). We note that for the easy-plane phase the ground state transverse magnetization depends on $q$ as $|F_\perp| = n_0 \sqrt{1 - (q/q_0)^2}$. Results (a), (c) are calculated at $t = 3770t_s$, while (b), (d) are calculated at $t = 1131t_s$. 

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4.3.2 Universal coarsening dynamics

The magnetization dynamics clearly show that the longitudinal (transverse) magnetization dominates at times sufficiently long after the quench for the easy-axis (easy-plane) quenches. As discussed in Sec. 4.2.1 this motivates us to define the order parameters

\[
\phi(r) = \frac{1}{n_0} F_z(r), \quad \text{Easy-axis,} \tag{4.4}
\]

\[
\phi(r) = \frac{1}{n_0} F_\perp(r), \quad \text{Easy-plane,} \tag{4.5}
\]

to characterize the development of order. Our interest in this chapter lies not in the emergence of local order (as characterized by the local magnetization), but in the evolution of the ordered domains on long time scales. In the language of Sec. 2.6, we are interested in the second stage of phase ordering dynamics rather than the first stage. Examples of these domains and their evolution are shown in Fig. 4.4 revealing the tendency of the domains to grow with time. These domains are described by the order parameter correlation function

\[
G(r, t) = \langle \phi(0) \cdot \phi(r) \rangle_t, \tag{4.6}
\]

where the average is taken at a time \( t \) after the quench. In practice we can calculate this correlation function utilizing the translational invariance of our
simulations to spatially average i.e. calculate

\[ G(r, t) = \frac{1}{l^2} \int d^2 r' \phi(r') \cdot \phi(r + r'), \quad (4.7) \]

and also use isotropy to perform an angular average over all points at a distance \( r \). To further improve statistical sampling we also average over eight simulation trajectories conducted with different initial noise.

The temporal evolution of the correlation function is shown in the insets to Figs. 4.5(a) and (b). The length scale over which the correlation function decays can be taken to define a characteristic domain size. As time progresses this length scale is seen to grow as order extends over large regions. As anticipated by the theory of phase ordering dynamics (see Sec. 2.6), we find that this growth exhibits dynamic scale invariance: correlations of the order parameter at late times collapse onto a single universal curve \( f(r) \) when lengths are scaled by a large characteristic length scale \( L(t) \), i.e.

\[ f(r) = G(r/L(t), t). \quad (4.8) \]

For the easy-axis phase, we take \( L(t) \) to be the first zero crossing of \( G(r, t) \). For the easy-plane phase, we take \( L(t) \) to be the point where \( G(r, t) = 0.25 G(0, t) \). Using these length scales we demonstrate the correlation function collapse in Figs. 4.5(a) and (b).

The growths of \( L(t) \) for the easy-axis quenches are shown in Fig. 4.5(c). Here we find that for a range of \( q \) values \( L(t) \sim t^{1/z} \) with \( z = 3/2 \). This is consistent with the dynamic critical exponent of a binary fluid in the inertial hydrodynamic regime [376]. The scale invariant dynamics can be ascribed to a process of hydrodynamic flow of the \( F_z \) superfluid velocity, see Sec. 4.3.4.

In the easy-plane phase, we find that \( L(t) \sim t/\ln(t/5t_s) \) for a range of \( q \) values, giving a dynamic critical exponent of \( z = 1 \) with a logarithmic correction, see Fig. 4.5(d). A dynamic critical exponent of \( z = 1 \) is consistent with the Model E universality class, which describes a 2D non-conserved order parameter coupled to a second conserved field [18, 19]. This fits our system well, where the second conserved field is \( F_z \), see Sec. 4.3.4 and [224]. Note that incorporating conservation of energy into model E gives model E', which also has \( z = 1 \) [18].

The logarithmic correction to scaling in the easy-plane phase is attributed to the presence of vortices, in analogy with the XY-model. Much work has shown that vortices in the XY-model slow the rate of coarsening and give rise to a logarithmic correction to scaling [43, 46, 47, 51, 52, 54, 378, 379] (see also [380, 381]) so that true dynamic scale invariance \( L(t) \sim t^{1/z} \) is only obtained after a very long time. In the easy-plane phase the order parameter
Figure 4.5: Collapse of the order parameter correlation function onto a single curve when lengths are scaled by the growing length scale $L(t)$ for (a) the easy-axis phase and (b) the easy-plane phase. The insets show the correlation functions before rescaling. (c) and (d) show the length scales $L(t)$ used for the correlation function collapse in (a) and (b), respectively. Lines indicate the best fit to these results. Data in (a) is for $q = -0.3$ and data in (b) is for $q = -1.2$. 
Figure 4.6: Polar-core vortices with $\kappa = 1$ (•) and $\kappa = -1$ (◊) in the (a) early stages and (b) later stages of the easy-plane coarsening dynamics in a quadrant of the full simulation. Transverse magnetization indicated as in Fig. 4.4, but with saturation reduced to make vortices clear. (c) Evolution of the spatial region indicated with a dashed box in (b). The dashed boxes in (c) identify vortex antivortex pairs that annihilate during the dynamics. (d) Total number of polar-core spin vortices ($N_{\text{vort}}$) as a function of time from a single quench simulation, demonstrating that the vortex density is proportional to $L(t)^{-2}$. Results in this figure are for a simulation with $g_n/|g_s| = 3$ [300].
supports polar-core vortices. These vortices consist of a phase winding in the (in-plane) magnetization around an unmagnetized core. The state of a polar-core vortex is

$$\psi_{\text{vort}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sin \beta e^{-i\theta} \\ \sqrt{2} \cos \beta \\ \sin \beta e^{i\theta} \end{pmatrix},$$

(4.9)

where far from the vortex core $\cos \beta = \sqrt{(1 + q/2|g_s|n)/2}$ [335]. The in-plane magnetization angle $\theta$ rotates by $2\pi \kappa (\kappa \in \mathbb{Z})$ around the vortex centre. These vortices are known as polar-core vortices because the particle density is in the $\psi_0$ component at the centre of the vortex, to avoid the phase singularity in the $\psi_{\pm 1}$ components. We only observe polar-core vortices of charge $\kappa = \pm 1$ during the coarsening regime, noting that higher values of $|\kappa|$ are unstable. We find that the domain growth is associated with the annihilation of these vortices, Fig. 4.6(a)-(c). Furthermore, the number of vortices is correlated with $L(t)$: as $L(t)$ grows, the density of vortices and therefore the total vortex number decay as $L(t)^{-2}$, see Fig. 4.6(d).

A phase winding of the spin angle $\theta$ corresponds to a circulation of the $F_z$ superfluid current, since $F_z \propto \nabla \theta$ [367]. It is feasible to also have circulation in mass and other spin currents, which would give rise to other types of vortices. We only observe polar-core spin vortices during the coarsening regime.

We note that the exponents obtained in Fig. 4.5(a) and (d) vary slightly with $q$. This is consistent with finite-size effects [217, 382] and statistical sampling over the time range we can simulate. For the easy-axis quench we obtain a range of $1/z = 0.66 - 0.70$. For the easy-plane quench we obtain a range of $1/z = 0.98 - 1.01$. A fit of the form $L(t) t^{1/z}$ also fits the numerical data in Fig. 4.5(d) well, and yields $1/z = 0.71 - 0.78$, depending on $q$. To differentiate this fit from one with $z = 1$ and a logarithmic correction would require simulation times $t > 10^4 t_s$, which is well beyond the range of our simulations. Furthermore, including a logarithmic correction in the easy-plane quench shrinks the range of exponents obtained to $1/z = 0.98 - 1.01$, consistent with universal scaling. For each value of $q$, the standard error of the mean of $1/z$, computed from the eight simulation trajectories, is $\approx 0.01$, so this may also account for the observed variation in $1/z$ with $q$.

### 4.3.3 Fractal dimension of domains

The order parameter domains in Fig. 4.4 are separated by domain boundaries. By examining the correlation function (4.6) at length scales $r < L(t)$, we are
Figure 4.7: Plots of structure factor for (a) the easy-axis phase and (b) the easy-plane phase. For $kL > 1$ the plots exhibit a longwavelength Porod tail. The $k^{-3}$ tail in the easy-axis phase indicates the presence of smooth domain boundaries, whereas the $k^{-2.4}$ tail in the easy-plane phase is indicative of a fractal domain boundary.

able to extract information about these boundaries. In particular, we can determine the dimension of the domain boundaries, which can be non-integer if the boundary has a fractal structure. It can be shown that the small $r/L$ behaviour of the correlation function behaves as [383]

$$1 - G(r, t) \sim \left( \frac{r}{L(t)} \right)^{D - D_b}$$

(4.10)

where $D_b$ is the fractal dimension of the domain boundary and $D$ is the system dimension. For the smooth case in two dimensions, i.e. $D = 2$, $D_b = 1$, Eq. (4.10) reflects the intuitive result that the probability that two points a distance $r$ apart will lie in opposite domains is $\sim r/L$ for $r \ll L$. In general, the probability that two points a distance $r$ apart will lie in opposite domains is $\sim (r/L)^{D - D_b}$ for $r \ll L$ from which one can derive Eq. (4.10).

In principle it is possible to extract the small $r/L$ behaviour directly from the correlation functions in Figs. 4.5(a) and (b). Equivalently, we instead choose to examine the order parameter structure factor, which is the Fourier transform of the correlation function,

$$S(k, t) = \int d^2r \ G(r, t)e^{ik\cdot r} = L^2 \hat{f}(kL(t)).$$

(4.11)
The scaling form follows from setting $G(r,t) = f\left(r/L(t)\right)$, with $\hat{f}$ being the Fourier transform of $f$. Behaviour of the correlation function at length scales $r < L(t)$ then appears at high wavenumbers $kL(t) > 1$. Fourier transforming the result (4.10) results in a high wavenumber “Porod tail” in the structure factor [383–385],

$$S(k) \sim k^{-2D+D_b}.$$  (4.12)

Results for the structure factor for the easy-axis and easy-plane quenches are shown in Fig. 4.7(a) and (b), respectively. For the easy-axis case we observe a “knee” in the structure factor at $kL \sim 1.3$ followed by a Porod tail $S(k) \sim k^{-3}$ for $L > k^{-1} \gg \xi_s$ that indicates the presence of smooth domain walls i.e. $D_b = 1$. We also observe a Porod tail for the easy-plane case, but with a non-integer exponent, $S \sim k^{-2.4}$. This suggests that the easy-plane domain boundaries are fractal, with a dimension of $D_b \approx 1.6$.

To provide further evidence for this result, we determine a box-counting dimension for the domain boundaries directly. The box-counting dimension is defined through

$$d_b = -\lim_{l_b \to 0} \frac{\log N_b}{\log l_b}.$$  (4.13)

Here we cover the system with boxes of side length $l_b$ and count the number of boxes $N_b$ that contain a domain boundary. In the limit of small $l_b$ the slope of $\log N_b$ versus $\log l_b$ gives the box-counting dimension. This naturally connects with the probabilistic interpretation of the dimension we used to discuss result (4.10). In the easy-axis phase the domain boundaries can be identified by looking for non-zero gradients in the sign of $F_z$, see Fig. 4.8(a),(b). For the easy-plane phase, a single domain is not as well defined because the order parameter changes continuously. However, the domain patterns in Fig. 4.4(b) do show clear regions of largely one colour. These regions will give rise to the flat peak in the structure factor in Fig. 4.7(b) that occurs at length scales $kL < 1$. To extract boundaries between these regions, we choose a $\pi/5$ range of (in-plane) spin directions and define discrete domains of spins that lie in this range. By assigning a 1 to spins within the domain and a -1 to spins outside the domain we can identify domain boundaries in an analogous way to the easy-axis phase, see Fig. 4.8(c),(d). 1 Once domain boundaries have been identified, we can perform a box counting algorithm to

1 Alternatively, these “boundaries” can be interpreted as lines of constant transverse spin angle (“isolines”). A region of spin angle $\theta_i < \theta < \theta_i + \pi/5$ ($\theta_i \in [0, 2\pi)$) can be bounded by either a single isoline of spin angle $\theta_i$ or $\theta_i + \pi/5$, or two isolines of spin angle $\theta_i$ and $\theta_i + \pi/5$ that meet at a spin vortex.
Figure 4.8: Order parameter domains (a),(b) with the boxed region enlarged (c),(d) to show details of domain boundaries. The boundaries are marked by black dots. The axes scale in images (c) and (d) are the same, indicating that the easy-plane domain boundary has a much more convoluted structure than the smooth easy-axis domain boundary. For the easy-axis (easy-plane) phase results displayed are for $q = -0.3q_0$ ($q = 0.3q_0$) and $t = 3770t_s$ ($t = 1131t_s$).
Figure 4.9: Results for $N_b$ versus $l_b$ obtained from the box-counting algorithm. In the small $l_b$ limit, the plots reveal a box-counting dimension for the domain boundaries of (a) $d_b = 1.0$ for the easy-axis phase and (b) $d_b = 1.5$ for the easy-plane phase. The vertical lines on the horizontal axes mark the size of domains extracted from the spatial correlation function of the order parameter. For $l_b \gtrsim L(t)$, the slope changes to 2, since boxes larger than the domain size will likely contain a domain boundary. For the easy-axis (easy-plane) phase results displayed are for $q = -0.3q_0$ ($q = 0.3q_0$) and $t = 3770t_s$ ($t = 1131t_s$).
determine the box-counting dimension of the boundaries. We do this over an
order of magnitude of box sizes, which yields a box counting dimension of
d_b = 1.0 for the easy-axis domain boundaries, see Fig. 4.9(a). In comparison,
we obtain a box counting dimension of d_b ≈ 1.5 – 1.6 for the easy-plane
domain boundaries, Fig 4.9(b). For the easy-plane phase, we can repeat the
box-counting algorithm for different domains of spin range, which give results
consistent with Fig. 4.9(b). Both the easy-axis and easy-plane box counting
dimensions agree with the slopes of the Porod tails in Fig. 4.7. We note that
the Porod tail in the easy-plane phase is not accounted for by topological
defects (polar-core spin vortices), which would result in a k^{-4} tail [37, 41, 51]
(this will be explored more in Chap. 6).

4.3.4 Analytic models of coarsening

Simple analytic models can be used to obtain the dynamic critical exponents
found numerically in the previous section. The models describe a dynamic
process that is expected to be important in the coarsening dynamics. Imposing
dynamic scale invariance on the equation describing this process allows one
to extract a dynamic critical exponent.

Easy-axis phase

The dynamic critical exponent obtained in the easy-axis phase suggests that
inertial hydrodynamics is important in the coarsening [38, 376]. We derive this
inertial hydrodynamic process by considering the hydrodynamic formulation
of a spin-1 condensate [367] (see Sec. 3.9). Ferromagnetic condensates support
both mass and spin superfluid currents. We assume a condensate with constant
number density n and zero population in the m = 0 spin level. Note that
these conditions still allow for spatial variation in |F_z|, as long as the sum of
populations in the m = ±1 levels is constant. The F_z superfluid current is
then
\[ v_{f_z} \equiv \frac{\hbar}{2M_i} \left( \psi^\dagger f_z (\nabla \psi) - (\nabla \psi)^\dagger f_z \psi \right) = \frac{F_z}{n} v_M, \] (4.14)
where \( v_M \) is the mass superfluid velocity,
\[ v_M \equiv \frac{\hbar}{2M_i} \left( \psi^\dagger (\nabla \psi) - (\nabla \psi)^\dagger \psi \right). \] (4.15)
The current \( v_{f_z} \) can be understood through the continuity equation (valid
when \( F_z \) is conserved),
\[ \frac{\partial F_z}{\partial t} + \nabla \cdot n v_{f_z} = 0. \] (4.16)
Eq. (4.14) shows that the order parameter is transported by the mass current. The equation of motion for $v_M$ is

$$\frac{\partial v_M}{\partial t} + (v_M \cdot \nabla) v_M = -\frac{g_s}{2Mn} \nabla F_z^2. \quad (4.17)$$

We have omitted third order derivative terms in Eq. (4.17). The scaling properties of our system arise from gradual, large length scale processes for which third order derivatives will be small. Equation (4.17) takes the form of the Euler equation from fluid dynamics, with the term $g_s F_z^2 / 2Mn$ in place of the pressure. This term is in fact a pressure, as can be seen by considering the energy of regions $A$ across which $F_z$ changes little

$$E(A) = \frac{g_n}{2} n^2 A + qn A + \frac{g_s}{2} F_z^2 A. \quad (4.18)$$

The pressure in such a region is

$$P(A) = -\left( \frac{\partial E}{\partial A} \right)_{N,M_z} = \frac{g_n}{2} n^2 + \frac{g_s}{2} F_z^2 \quad (4.19)$$

where the partial derivative is evaluated for fixed atom number and fixed total $z$ magnetization $M_z = \int d^2x F_z$. With constant number density the gradual spatial variation of $P(A)$ arises only from the term $g_s F_z^2 / 2$. Equation (4.17) can therefore be written as

$$\frac{\partial v_M}{\partial t} + (v_M \cdot \nabla) v_M = -\frac{1}{Mn} \nabla P. \quad (4.20)$$

In mechanical equilibrium, the pressure difference, $\Delta P$, across a curved surface is related to the surface tension, $\sigma$, of the surface through the *Young-Laplace Equation* \[386\]

$$\Delta P \sim \frac{\sigma}{R}. \quad (4.21)$$

where $R$ is the curvature of the surface. This relationship arises because the excess surface energy in a curved surface gives rise to a force on this surface, and in equilibrium this must be balanced by a pressure difference. In the case of a condensate with two components separated by a domain wall, the surface energy arises from the kinetic energy across the wall \[387\]. We expect the balance of surface tension and pressure to be fast in comparison to the slow hydrodynamics driving domain growth, so that we can use Eq. (4.21) in Eq. (4.20).

We now want to impose dynamic scale invariance on Eq. (4.20). We carry out a scaling transformation of this equation, rescaling times by $T$, i.e.
\[ t \to t/T, \quad \text{and lengths by } L(T) \]. We note that \( \Delta P \) scales as \( L \) [from Eq. (4.21)] so that \( \nabla P \) scales as \( L^2 \). This gives

\[
\frac{T^2}{L} \frac{\partial v_M}{\partial t} + \frac{T^2}{L} (v_M \cdot \nabla) v_M = -L^2 \frac{1}{Mn} \nabla P, \tag{4.22}
\]

Imposing dynamic scale invariance amounts to Eq. (4.20) being preserved under this transformation. This can only occur if \( L(T) = T^{2/3} \). To write this in the usual scaling form \( L(t) \sim t^{1/z} \) we choose \( T \propto t \), giving \( z = 3/2 \) [37, 217, 376].

**Easy-plane phase**

For systems in the model E dynamic universality class, the dynamics of spin-waves dynamically coupled to a second conserved field are argued to be the important process during coarsening [19, 388, 389]. Here *dynamically coupled* means that there is no direct coupling between these two fields in the Hamiltonian, but there is a non-vanishing Poisson bracket relation between them that leads to coupling in the dynamic equations.

To show how spin waves arise in our system we begin with a variational ansatz [296]

\[
\psi(x, t) = \sqrt{n_0} \begin{pmatrix}
\sin \beta e^{-i\theta} \cos \left( \frac{\pi}{4} + \chi \right) \\
\cos \beta \\
\sin \beta e^{i\theta} \sin \left( \frac{\pi}{4} + \chi \right)
\end{pmatrix}, \tag{4.23}
\]

with \( \cos(2\beta) \equiv q/q_0 \) and variational parameters \( \theta(x, t) \) and \( \chi(x, t) \). In the ground state we would have \( \chi = 0 \) and uniform \( \theta \). Spatial variation in \( \theta \) corresponds to fluctuations in the direction of transverse spin and will give rise to spin waves. Non-zero \( \chi \) describes fluctuations in the (conserved) \( F_z \) magnetization.\(^2\)

We assume fluctuations of \( \chi \) are small and so consider fluctuations up to quadratic order in \( \chi \) only. The Hamiltonian then takes the form

\[
H = \int d^2 x \left[ \frac{\hbar^2 n_0 \sin^2 \beta}{2M} \left( |\nabla \theta|^2 + |\nabla \chi|^2 \right) - 2g_s n_0^2 \sin^2 \beta \cos(2\beta) \chi^2 \right]. \tag{4.24}
\]

\(^2\)In the ansatz (4.23), \( E_k \sim c_n \hbar k \), with \( c_n = \sqrt{g_s n_0/M} \) [358]. Long wavelength gapped modes will have higher energy and therefore faster dynamics than the gapless modes in (4.23). The gapped modes will therefore be less important in the slow coarsening dynamics. Modes with a steep spectrum will also have higher energy at any given \( k \), and lower occupation.
The Lagrangian is obtained through a Legendre transformation of the Hamiltonian with respect to the conjugate variables $\psi$ and $i\psi^\dagger$. This gives

$$L = \Im \left( \psi^\dagger \dot{\psi} - H \right)$$

[296, 390]. To second order in $\chi$ and $\dot{\theta}$ we obtain

$$L = 2n_0 \sin^2 \beta \int d^2 x \left[ \chi \dot{\chi} - \frac{\hbar^2}{4M} \left( |\nabla \chi|^2 + |\nabla \theta|^2 \right) + g_s n_0 \cos(2\beta) \chi^2 \right]. \quad (4.25)$$

Formulating the problem in this form gives the conjugate variable relations

$$\frac{\delta L}{\delta \dot{\theta}(x)} = 2n_0 \sin^2 \beta \chi(x), \quad \frac{\delta L}{\delta \dot{\chi}(x)} = 0. \quad (4.26)$$

The first of these relations reflects the dynamic coupling between fluctuations of the direction of the order parameter and fluctuations of the conserved field $F_z$. This relation also reflects that conservation of $F_z$ magnetization is connected with rotational symmetry about the $F_z$ axis. Evaluating Lagrange’s equations (4.26), which decouple by taking a second time derivative, gives

$$\ddot{\theta} = q_0 \cos(2\beta) \frac{\hbar^2}{2M} \nabla^2 \theta - \frac{\hbar^4}{4M^2} \nabla^4 \theta, \quad (4.27)$$

$$\ddot{\chi} = q_0 \cos(2\beta) \frac{\hbar^2}{2M} \nabla^2 \chi - \frac{\hbar^4}{4M^2} \nabla^4 \chi. \quad (4.28)$$

The spin-wave fluctuations ($\theta$) and $F_z$ fluctuations ($\chi$) can be expanded in Fourier modes, giving a spectrum

$$E_k = \sqrt{\frac{\hbar^2 k^2}{2M} \left( q_0 \cos(2\beta) + \frac{\hbar^2 k^2}{2M} \right)}. \quad (4.29)$$

For the long wavelength excitations ($k^2 \ll 2Mq_0 \cos(2\beta)/\hbar^2$), Eq. (4.27) gives the equation of motion for spin waves

$$\ddot{\theta} = q_0 \cos(2\beta) \frac{\hbar^2}{2M} \nabla^2 \theta. \quad (4.30)$$

Scaling lengths by $L$ and times by $T$ in Eq. (4.30), and setting $T \propto t$, gives scaling $L \sim t^{1/z}$, with $z = 1$.

We could alternatively approach the coarsening from the perspective of polar-core spin vortices, Eq. (4.9). These are not accounted for in the Fourier expansion. 

\[ A comparison of the Lagrangian (4.25) with Eq. (2.51) shows that $F_z$ plays a similar role to the response field $\varphi$. \]

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Lagrangian (4.25), which considers quadratic fluctuations only. A simple
model for the dynamics of polar-core spin vortices was given in [268],
\[
m\ddot{R}_i = \frac{\hbar^2 n_0 \sin^2 \beta}{M} \sum_{j \neq i} \frac{\kappa_i \kappa_j}{|R_i - R_j|^2} (R_i - R_j),
\]
where \( R_i \) denotes the position of spin vortex \( i \) with charge \( \kappa_i \) and \( m \) is the
“mass” of a spin vortex. This model arises by noting that a spin vortex is
composed of two scalar vortices in the \( m = \pm 1 \) levels. Scalar vortices in the
\( m = 1 \) level interact according to the usual scalar vortex dynamics [391–393],
and similarly for vortices in the \( m = -1 \) level. However, there is also an
attractive interaction between the \( m = \pm 1 \) vortices within a single spin vortex,
which arises from the spin-spin interaction in (4.1). This contributes a crucial
core energy that leads to the second order equation of motion, Eq. (4.31).
This differs from the first order equation of motion that arises in scalar vortex
dynamics [392], which would give rise to a dynamic critical exponent of \( z = 2 \).
For more details on the dynamics of polar-core spin vortices, see [268]. We
will discuss polar-core spin vortex dynamics in more detail in Chap. 7.

If we assume that Eq. (4.31) is invariant under a rescaling of lengths
by \( L \) and times by \( T \propto t \), we again obtain the exponent \( z = 1 \). However,
the interaction between spin waves and polar-core spin vortices will likely
change both the spin wave dynamics (4.30) and the vortex dynamics (4.31).
In the \( XY \)-model, it is necessary to couple the vortices to damping degrees
of freedom to justify the logarithmic correction to scaling [43]. It may be
possible to extend Eq. (4.31) to allow coupling to spin waves [394], which
may reveal a logarithmic correction to scaling.

4.3.5 Thermalisation of excitations in the easy-axis phase

As the domains coarsen, energy is liberated into excitations on top of the
ordered phase. Over time we expect that these excitations will thermalise. The
long wavelength coarsening of the order parameter should be slow compared
to other thermalisation processes. Therefore by the time the order parameter
domains are large, other modes in the system will have thermalised. We can
test this thermalisation by examining the population of Bogoliubov modes on
top of the ground state. The Bogoliubov modes may depend on the orientation
of the ground state order parameter and so may change across the system
due to the presence of domains. Also when the Bogoliubov modes and the
condensate occupy the same \( m \) sublevels it is difficult to distinguish between
them. Fortunately for the easy-axis case the condensate only occupies the
\( m = \pm 1 \) sublevels, while there is a spin-wave branch that occupies the \( m = 0 \)
sublevel and is insensitive to the orientation of the order parameter, i.e. not affected by the presence of domains [357]. We can therefore determine the population distribution in the $m = 0$ mode and compare this to the value expected in equilibrium.

Since we perform classical field simulations of the quench, we expect to see excitations on top of the ground state populated according to the equipartition theorem when the system is in equilibrium. As will be shown below, we find that the temperature of the $m = 0$ modes is large compared to their energy, so that the modes are highly occupied and equipartition is valid. For the spin-1 system there are three Bogoliubov branches on top of the ground state [357, 358]. If energy is distributed amongst all the modes of these branches according to equipartition, we would obtain a total energy of $3N^{2}k_{B}T$, where $N^{2}$ is the total number of grid points (i.e. spatial modes) in the numerical simulation, $T$ is the temperature of the modes and $k_{B}$ is Boltzmann’s constant. Equating this to the total energy liberated from the quench [Eq. (4.3)], we obtain the temperature,

$$k_{B}T = \frac{1}{3}n_{0} \left( \frac{l}{N} \right)^{2} \left( \frac{q_{0}}{4} - q \right).$$

(4.32)

The $m = 0$ Bogoliubov modes (spin-waves) have energy [357]

$$E_{k} = \epsilon_{k} - q,$$

(4.33)

with $\epsilon_{k} \equiv \hbar^{2}k^{2}/2M$. According to the equipartition theorem, we expect the population of an $m = 0$ mode with energy $E_{k}$ to be

$$n_{k} = \frac{k_{B}T}{E_{k}}.$$

(4.34)

Following Eqs. (4.33) and (4.34) we fit late time numerical data for $1/n_{k}$ to the functional form $a + bk^{2}$, with $a$ and $b$ as fitting parameters. From $a$ and $b$ we can extract the temperature $k_{B}T$ and the fitted energy gap $E_{\text{gap}}$. Figure 4.10(a) shows $1/n_{k}$ data for quenches to $q = -0.3q_{0}$, $q = -0.6q_{0}$, $q = -0.9q_{0}$ and $q = -1.2q_{0}$. The $a + bk^{2}$ fits the data well, showing that energy in the $m = 0$ level is equipartitioned amongst the momentum modes. The small $k$ behaviour is shown in Fig. 4.10(b) and agrees well with the predicted energy gap of $-q$ [from Eq. (4.33)]. The extracted temperatures are shown in Fig. 4.10(c), along with the prediction (4.32). The agreement between the fitted temperature and predicted temperature is good.

This temperature is dependent on the number of simulation grid points, and therefore its value is not directly comparable to experiments. A more accurate consideration of temperature could be obtained by utilising the full stochastic classical field methods, as presented in [328, 354].

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Figure 4.10: (a) Inverse population of the $m = 0$ level across momentum modes for quenches to four different easy-axis $q$ values at time $t = 3770t_s$. Dots are numerical data for a single simulation, averaged over azimuthal angle. Lines are best fits of the form $1/n_k = a + bk^2$ where $a$ and $b$ are fitting parameters. (b) Enlarged small $k$ behaviour of (a) (note change of vertical axis). The data agrees well with Eq. (4.34), which predicts that the vertical axis intercept should be $-q/q_0$. (c) Temperature of the $m = 0$ excitations versus $q$. Circles are fits from (a), solid line is the theoretical prediction assuming equipartition, Eq. (4.32).
Figure 4.11: (a) Inverse population of the $m = 0$ level across momentum modes at different times following the quench. Dots are numerical data for a single simulation, averaged over azimuthal angle. Lines are best fits of the form $1/n_k = a + bk^2$ with $a$ and $b$ as fitting parameters. The linear nature of the data shows that energy is equipartitioned amongst the (gapped) free particle modes. The slope decreases with time, reflecting an increase in temperature of the $m = 0$ excitations. For times $t \lesssim 80t_s$ (not shown) energy deviates from being equipartionated, coinciding with times where $\langle |F_z|^2 \rangle < \langle |F_\perp|^2 \rangle$. Data is for $q = -0.3q_0$. (b) Circles show growth of the temperature of the $m = 0$ excitations, extracted from the fit to the numerical data in (a). Dashed line shows temperature computed from (4.32). (c) Circles show the energy gap in the spectrum extracted from the fit to the numerical data in (a) versus time. At late times the gap approaches the value $-q$, marked by the dashed line, in agreement with Eq. (4.33).
We also determine the rate that energy flows to the $m = 0$ Bogoliubov modes by examining how the mode populations change with time, see Fig. 4.11(a). The extracted temperature and energy gap are shown in Figs. 4.11(b) and (c) respectively. Even for quite early times, $t \approx 80t_s$, energy is equipartitioned amongst the momentum modes.\footnote{The earliest data in Fig. 4.11(a) ($t = 81t_s$) shows equipartition amongst higher momentum modes ($\epsilon_k \gtrsim 4q_0$), which are at a lower temperature than the modes with lower momentum at this time. This may be because lower momentum modes couple more effectively to the condensates in the $\psi_{\pm 1}$ spin levels, leading to more rapid thermalisation between the spin levels at this momentum scale.} The temperature, however, does not equilibrate until a time $t \sim 500t_s$. The flow of energy to the $m = 0$ level is therefore slower than equilibriation amongst the $m = 0$ momentum modes. For times $t \lesssim 80t_s$ (not shown) energy deviates from being equipartitioned, coinciding with times where $\langle |F_z|^2 \rangle < \langle |F_{\perp}|^2 \rangle$, see Fig. 4.2(a).

The growth of temperature in Fig. 4.11(c) matches the growth of $F_z$ magnetization in the system, see Fig. 4.12. The time scale of growth also matches the time scale of decay of transverse magnetization. We can determine this time scale by fitting the growth to the functional form $\propto (1 - e^{-t/\tau})$. This empirical fit gives a growth rate of $\tau \approx 150t_s$. We note that the density interaction $g_{nn'} \sigma^2$ will likely drive the thermalisation of the $m = 0$ momentum modes, whereas the changes in spin density are driven by the smaller spin interaction.

### 4.4 Conclusions and Outlook

In this chapter we have examined the quench dynamics of a quasi-two-dimensional ferromagnetic spin-1 condensate. We have found that for quenches to $q < 0$ (into the easy-axis ferromagnetic phase), order grows with a dynamic critical exponent of $z = 3/2$ (also see [217, 228, 231]). For quenches to $q$ in the range $0 < q < q_0$ (into the easy-plane ferromagnetic phase) we find that order grows with a dynamic critical exponent of $z = 1$. With our numerical results we have verified that the late time coarsening dynamics is scale invariant by demonstrating correlation function collapse when scaling by $L(t)$. For the easy-plane quench we demonstrate the important role of polar-core vortices by showing that the number of vortices scales as $L(t)^{-2}$ during the coarsening dynamics. Thus we can interpret easy-plane coarsening as occurring via the mutual annihilation of vortex anti-vortex pairs.

To provide insight into the origin of the growth of order we have discussed simple analytic models that capture the essential dynamics of the coarsening, and correctly predict the dynamic critical exponents. For the easy-plane...
Figure 4.12: Growth of temperature (black circles, left axis) and the mean squared $F_z$ magnetization $\langle F_z^2 \rangle / n_0^2$ (plain red line, right axis) versus time. The black dashed line shows a fit to the temperature growth of the form $\propto (1 - e^{-t/\tau})$, where $\tau \approx 150 t_s$ gives the growth rate. The red line with dots shows the decay of the mean squared transverse magnetization $\langle F_\perp^2 \rangle / n_0^2$ (right axis). Mean squared magnetizations are calculated as described in Fig. 4.2
phase we can develop such a model based either on spin-wave or vortex dynamics, although we are unable to analytically obtain the log corrections to the growth law, which remains an area for future investigation.

The structure of the ordered domains was studied by quantifying the Porod tails in the order parameter structure factor, and by a direct spatial analysis using the box counting algorithm. Both approaches show that the easy-axis domain walls are regular with a dimension of $D_b = 1$, whereas the easy-plane domain walls are fractal with a dimension of $D_b \approx 1.6$. Possible physical implications of this fractal structure includes diffusion limited aggregation [395], or Schramm (stochastic)-Loewner evolution and the associated conformal invariance [396]. We also note recent work considering the domain size distribution and domain wall percolation in binary condensates [220, 221], which would be an interesting direction to pursue for the spinor system.

We have also considered how the energy liberated from the quench rethermalizes in the system. The deeper the quench more energy is liberated and the system exhibits larger fluctuations. In order to quantify the thermalization in the easy-axis quench we demonstrate an analysis technique that allows us to measure the temperature and distribution of spin-waves. Using this we show that local equilibrium is established in the spin waves on moderate time scales, but continues to evolve as the order parameter domains anneal.

Finally we discuss the requirements that must be met to observe coarsening in experiments. First, the the size of the condensate must be much larger than the spin healing length $\xi_s$. With the development of flat optical traps (see [195, 196]) this condition is easily met. Furthermore, these flat traps minimise inhomogeneous effects present in harmonic trapping potentials, and appear ideally suited to studies of phase transition dynamics. The second requirement is that the system has a sufficiently long lifetime that the coarsening dynamics can be monitored over timescales much longer than the spin time $t_s$. In $^{87}$Rb this time is typically $t_s \sim 100$ ms and experiments have been able to study coarsening dynamics for times up to 4 s [295]. Our results suggest that such time scales may be sufficient to observe the beginning of scale invariant domain growth in the easy-plane phase. In the easy-axis phase the onset of scale invariant coarsening takes longer, however recent results show that this can be accelerated by quenching from an antiferromagnetic (rather than polar) initial state [377].
Chapter 5

Coarsening dynamics in the isotropic phase

In zero magnetic field (i.e. zero quadratic Zeeman energy) the ground state manifold of a ferromagnetic spin-1 condensate is SO(3) and exhibits $\mathbb{Z}_2$ vortices as topological defects. In this chapter we investigate the phase ordering dynamics of this system after being quenched into this isotropic ferromagnetic phase from a zero temperature unmagnetized phase. Following the quench, we observe the ordering of both magnetic and gauge domains. We find that the growth of these domains exhibits dynamic scale invariance, with a size that scales with time as $L(t) \sim t^{1/2}$. The coarsening dynamics progresses as $\mathbb{Z}_2$ vortices annihilate, however we find that at finite energy a number of these vortices persist in small clumps without influencing magnetic or gauge order. We consider the influence of a small non-zero magnetic field, which reduces the ground state symmetry, and show that this sets a critical length scale such that when the domains reach this size the system dynamically transitions in order parameter and scaling behaviour from an isotropic to an anisotropic ferromagnetic superfluid.

5.1 Introduction

Bose-Einstein condensates provide an isolated quantum system with a tractable microscopic description to explore coarsening dynamics. In the previous chapter we utilised this, identifying universal phase ordering dynamics in both the easy-axis and easy-plane ferromagnetic phases. Both phases revealed phase ordering consistent with classical dynamic universality classes [19], as have studies of immiscible binary condensates [217]. A unique feature of quantum systems is gauge symmetry, and this opens up the possibility to explore the
Figure 5.1: Ground state phase diagram of a ferromagnetic spin-1 condensate as a function of the quadratic Zeeman energy $q$. The net $F_z$ magnetization is assumed to be 0. For large $q$, the condensate is unmagnetized (polar phase). For $q < q_0 \equiv 2|g_s|n_0$, the system magnetizes. The direction of magnetization is: along $z$ for $q < 0$, termed easy-axis; isotropic for $q = 0$; and in the $xy$-plane for $0 < q < q_0$, termed easy-plane. The choice of spin directions in each phase is shown by arrows in the respective spin spheres. The red arrow indicates an instantaneous quench from deep in the polar phase to $q = 0$.

phase ordering of ground state manifolds not available in classical systems. Exploring the long time phase ordering in such systems is an area that has received little attention. Although coarsening dynamics in systems with U(1) gauge invariance has been explored [35, 187, 377, 397], this manifold is shared by a classical XY spin system. The combination of spin and gauge symmetries in spinor condensates offer order parameter manifolds and associated topological defects that cannot be found, or at least are difficult to construct, in classical systems.

The ground state manifold of an isolated ferromagnetic condensate is SO(3), arising from the spin and gauge invariance of the system. Symmetry breaking in this phase has been observed in the spin-1 case following a temperature quench [295]. A quantum quench from the unmagnetised polar phase to the SO(3) phase can also be induced by quenching the quadratic Zeeman field, see Fig. 5.1. Coarsening dynamics in an SO(3) system has recently been explored in an antiferromagnetic lattice system, but such systems exhibit geometric frustration that suppresses the coarsening [360]. Apart
from this, coarsening dynamics in an SO(3) system has to our knowledge not
been explored. An SO(3) system supports $\mathbb{Z}_2$ vortices that we can expect to
be present during the coarsening dynamics. Theoretical studies have shown
that $\mathbb{Z}_2$ vortices can be stabilised in rotating condensates [261, 265, 398],
however much less work has explored their role in dynamical processes. In
addition, whether or not a topological phase transition exists between bound
and unbound $\mathbb{Z}_2$ vortices has drawn interest in frustrated antiferromagnetic
lattices [399, 400], but so far results are inconclusive.

In this chapter we explore coarsening dynamics in the SO(3) phase of a
ferromagnetic spin-1 condensate following a quench from the polar phase to
the isotropic phase. The quench is implemented by a sudden change in $q$
from a large positive energy to $q = 0$, see Fig. 5.1. We find that coarsening of
both spin and gauge domains occur, and that both ordering processes exhibit
dynamic scale invariance with a critical exponent of $z = 2$. We find that the
quench generates many $\mathbb{Z}_2$ vortices that initially decay in a way consistent
with the formation of order, but for longer times decay much slower with a
 persistence of closely “bound” vortices. We also explore how the coarsening
dynamics behaves for small but non-zero $|q|$ and identify a transition between
two different dynamic universality classes that occurs when the domains grow
larger than a critical size.

The outline of this chapter is as follows. In Sec. 5.2 we discuss the ground
state manifold of the isotropic phase. Our results on phase ordering in the
isotropic phase are presented in Sec. 5.3, where we show universal phase
ordering across the full SO(3) manifold. We identify $\mathbb{Z}_2$ vortices, and relate
these to the phase ordering that we observe. In addition we explore phase
ordering for quenches to small but nonzero $q$, observing a transition from
isotropic to anisotropic order parameter and scaling, which we find occurs at
a critical domain size. We conclude in Sec. 5.4, and discuss the outlook of
future work that could be explored in this spinor phase.

5.2 System

The ground state magnetic phases of a ferromagnetic spin-1 condensate for
varying $q$ are shown in Fig. 5.1 (see Sec. 3.7 for a thorough discussion). Here
we are interested in the phase ordering near $q = 0$. The ground state spinor
at $q = 0$ can be written as [338]

$$
\psi = \sqrt{n_0} e^{i\alpha} \begin{pmatrix}
\frac{e^{-i\theta} \cos^2 \frac{\beta}{2}}{1 \sqrt{2}} \\
\frac{1}{\sqrt{2}} \sin \beta \\
\frac{e^{i\theta} \sin^2 \frac{\beta}{2}}{1 \sqrt{2}}
\end{pmatrix},
$$

(5.1)
where $\alpha \in [0, 2\pi)$ is the phase of the spinor component $\psi_0$, while $\theta \in [0, 2\pi)$ and $\beta \in [0, \pi]$ determine the direction of spin density

$$F = n_0(\sin \beta \cos \theta, \sin \beta \sin \theta, \cos \beta).$$

Spatial variation of $\alpha$ gives rise to $\psi_0$ gauge domains, while spatial variation of $\theta$ and $\beta$ gives rise to magnetic domains.

The full symmetry of the ground state (5.1) is SO(3) [338]. Examining Eq. (5.1), we can see that a point on the ground state manifold can be specified by a single vector, where the direction of the vector corresponds to the direction of magnetization and the length of the vector (between $-\pi$ and $\pi$) corresponds to the global phase. This maps out a solid sphere of points, with the additional feature that antipodal points (i.e. points diametrically opposite each other on the sphere) correspond to the same point (since $e^{-i\pi} = e^{i\pi}$). Equivalently, this manifold can be represented by the 3-sphere $S^3$ with antipodal points identified, which is known as the real projective space $\text{RP}^3$. Such a manifold supports topological defects called $\mathbb{Z}_2$ vortices [265]. These occur when the order parameter around a loop in the physical system maps onto a path joining two antipodal points in $S^3$, which forms a loop in $\text{RP}^3$ that cannot be continuously undone.

The annihilation of topological defects is intimately linked with phase ordering [37]. We expect $\mathbb{Z}_2$ vortices to be produced by the quench and decay during the coarsening dynamics. The $\mathbb{Z}_2$ vortices couple the gauge and spin ordering, as a circulation in the gauge angle $\alpha$ can be continuously transformed into a circulation in the spin angle $\theta$ so that gauge defects and spin defects are not independent [335, 338]. This is in contrast to the easy-axis phase, where the gauge and magnetic degrees of freedom support distinct defects and the ordering of each is different [377].

## 5.3 Results

### 5.3.1 Coarsening dynamics

To simulate the quench dynamics we numerically evolve the spin-1 Gross-Pitaevskii equations (see Sec. 3.5) on a $1024 \times 1024$ grid with initial condition of a polar condensate $\psi = \sqrt{n_0}(0, 1, 0)^T$ that has vacuum noise added to Bogoliubov modes for $q_i = \infty$ according to the truncated Wigner prescription (see Sec. 3.11). The noise is necessary to seed the formation of symmetry breaking domains. We study the growth and coarsening of the order
parameters,

\[ \phi = \begin{cases} F/n_0, & \text{(spin)}, \\ \psi_0/\sqrt{n_0}, & \text{(gauge)}. \end{cases} \]  

To do this we examine the correlation functions

\[ G_\phi(r, t) = \langle \phi(r)\phi(0) \rangle_t - |\langle \phi(0) \rangle_t|^2, \]  

where the average is taken at a time \( t \) after the quench. We perform the average by invoking spatial and rotational invariance of correlations, and also average over four simulation trajectories conducted with different initial noise. We use a condensate density of \( n_0 = 10^4/\xi_s^2 \), where \( \xi_s \equiv \hbar/\sqrt{q_0 M} \) is the spin healing length. The system size is \( l \times l = 800\xi_s \times 800\xi_s \) and \( g_n/|g_s| = 10 \).

At \( t = 0 \) the quadratic Zeeman shift is instantaneously quenched to zero. The initial exponential growth of unstable modes resulting in a growth of the spin density \( F \) is well understood [227, 283, 286, 296] (see Sec. 3.10). For times \( t \gg t_s \equiv \hbar/q_0 \) this growth stabilises, see Fig. 5.2(a). Across these long evolution times coherent domains of spin and \( \psi_0 \) fluctuations coarsen.

We examine the growth of domains by plotting the correlation functions (5.4) for various times, see Fig. 5.2(b),(c). The spatial extent of correlations grow with time (insets) while the shape of the correlation function is preserved. This shows that the domain growth is scale invariant, consistent with the theory of phase ordering dynamics. Rescaling the position coordinate of the correlation functions \( G_\phi \) by the respective domain size \( L_\phi(t) \) results in the correlation functions at different times collapsing onto a single curve, where we take \( L_\phi(t) \) to be the shortest distance where \( G_\phi(r, t) = 0.2G_\phi(0, t) \).

We plot the growing length scales \( L_F(t) \) and \( L_\psi(t) \) in Fig. 5.2(d). We see that the domains grow as a power law with time \( \sim t^{1/z} \), and a fit to this growth gives \( z = 2 \).

For long wavelength (low energy) dynamics, the spin density \( F \) obeys a Landau-Lifshitz equation (LLE) modified due to the advection of the order parameter by the superfluid velocity [401]. Imposing that the LLE obeys dynamic scale invariance gives a dynamic critical exponent \( z = (d + 2)/2 \) for a \( d \) dimensional system at low temperatures [389] (see also [402]), consistent with our results.

### 5.3.2 \( \mathbb{Z}_2 \) vortices

In addition to long wavelength excitations, the state (5.1) supports topological defects known as \( \mathbb{Z}_2 \) vortices. The nontrivial topology of \( \mathbb{Z}_2 \) vortices comes
Figure 5.2: (a) Growth of mean squared magnetization and $\psi_0$ fluctuations $\delta \psi_0 \equiv \psi_0 - \langle \psi_0 \rangle$. Both quantities grow until $t \sim 500t_s$, and then remain steady. The final value of mean squared magnetization is less than the ground state value $n_0^2$ due to heating from the quench. The spatial correlation functions of (b) $F/n_0$ and (c) $\psi_0/\sqrt{n_0}$ [see Eq. (5.4)] during the coarsening dynamics. Insets: correlation functions for various times versus the spatial coordinate $r$. Main plots: correlation functions for various times versus the rescaled coordinates $r/L_F(t)$ for (b) and $r/L_0(t)$ for (c). Correlation function collapse onto a single curve shows the domain growth exhibits dynamic scale invariance. (d) The evolution of $L_F(t)$ and $L_0(t)$ versus time, showing that they grow as $t^{1/2}$.
about as follows (see also the discussion in Sec. 5.2). The total superfluid velocity for the state (5.1) can be calculated (see Sec. 3.9) to give \[335\]

\[ M \mathbf{v}_M / \hbar = \nabla \alpha + \nabla \theta - (1 - \cos \beta) \nabla \theta. \tag{5.5} \]

Single-valuedness of the state imposes that \( \oint_C d\alpha = 2\pi n_\alpha \), \( \oint_C d\theta = 2\pi n_\theta \) and \( \oint_C d\beta = 0 \) for any closed path \( C \), where \( n_\alpha \) and \( n_\theta \) are integers \[335\]. The difference between the superfluid velocity and Berry phase \((1 - \cos \beta) \nabla \theta\) is \( \nabla (\alpha + \theta) \), which is quantized. A state with even phase winding of \( \nabla (\alpha + \theta) \), i.e. even \( n_\alpha + n_\theta \), is homotopic to the vortex free state, while all odd phase windings are topologically equivalent to a single phase winding \[335, 338\]. The group of homotopically distinct topologies in the SO(3) system is therefore isomorphic to the group \( \mathbb{Z}_2 \), hence the vortex name.

The positions of \( \mathbb{Z}_2 \) vortices can be identified by points around which the phase winding of \( \alpha + \theta \) is odd. Figure 5.3 shows the decay of the total number of \( \mathbb{Z}_2 \) vortices versus time. Vortex annihilation occurs through the collision of two oddly charged vortices. For \( t \lesssim 10^3 t_s \), the number of \( \mathbb{Z}_2 \) vortices decays as \( t^{-1} \). This decay is consistent with the average distance between vortices growing as \( t^{1/2} \), i.e. the same scaling as the domain growth. Thus the annihilation of vortices appears associated with the coarsening dynamics. For \( t > 10^3 t_s \), the decay of \( \mathbb{Z}_2 \) vortices decreases substantially, even though the magnetic and gauge domains continue to grow. The residual \( \mathbb{Z}_2 \) vortices must therefore be present in small ("bound") groups with a net even charge, so as not to destroy the magnetic or gauge order. We expect these bound vortices arise out of heating from the quench. We have also carried out simulations using the damped spin-1 GPEs \[353, 403, 404\], which remove excess thermal energy and drives the system toward the vortex free ground state. In this case there is a continual \( t^{-1} \) decay of \( \mathbb{Z}_2 \) vortices [see Fig. 5.3], consistent with magnetic and gauge ordering (note that we obtain a \( t^{1/2} \) growth of magnetic and gauge domains in the damped regime). The bound \( \mathbb{Z}_2 \) vortices therefore annihilate when damping is included, as expected.

The phase winding of a \( \mathbb{Z}_2 \) vortex is made up of \( \alpha \) and \( \theta \) phase windings. We detect \( \mathbb{Z}_2 \) vortices by looking for such phase winding around plaquettes of size twice our simulation grid spacing. We find these occur predominantly as "spin vortices" with \( n_\theta = \pm 1 \), \( n_\alpha = 0 \) and "gauge vortices" with \( n_\alpha = \pm 1 \), \( n_\theta = 0 \). Spin vortices arise from a circulation of the spin vector after it is projected onto the plane perpendicular to the spin quantization axis; gauge vortices arise from a circulation in the phase of \( \psi_0 \), which can also be understood as a

\[1\]The angle \( \beta \) is confined between 0 and \( \pi \) (see Eq. 5.1) and therefore cannot complete a \( 2\pi \) phase winding.
Figure 5.3: Decay of the number of $\mathbb{Z}_2$ vortices (which can be subdivided into “spin” and “gauge” vortices) with time. Results are the average of four simulations. For $t \lesssim 10^3 t_s$, the number of $\mathbb{Z}_2$ vortices decays as $t^{-1}$, consistent with magnetic and gauge ordering. During this time the number of spin vortices is much larger than the number of gauge vortices. For $t > 10^3 t_s$, the decay of $\mathbb{Z}_2$ vortices decreases substantially leading to a significant number of residual “bound” vortices. Results for the number of $\mathbb{Z}_2$ vortices for a damped GPE simulation are also shown.
spin rotation around the direction of the spin vector.\footnote{If the spin vector aligns with the direction of quantization, this division is ambiguous. Since the whole spin sphere is accessible, such cases will be negligibly rare.} It is not true, however, that spin vortices affect only spin order and gauge vortices affect only gauge order. Figure 5.3 shows that throughout the simulation the number of spin vortices (at the length scale $\approx 1.6\xi_s$ of our vortex detection) is always much larger than the number of gauge vortices, even though the spin and gauge coarsening dynamics occur in sync (see Fig. 5.2). A vortex may consist of spin circulation at one length scale (say close to the core) but then be continuously transformed to a gauge vortex at larger length scales, and vice versa, so that spin and gauge vortices detected at one length scale can each affect both spin and gauge order at another length scale.

5.3.3 Coarsening for non zero $|q|$  

During the coarsening for $q = 0$, all of the components of $F$ exhibit dynamic scale invariance with a length scale that grows as $t^{1/2}$. In comparison, for the easy-axis phase with $q < 0$ only the $F_z$ correlations exhibit scale invariant growth but with a $t^{2/3}$ growth law, while for the easy-plane phase with $0 < q < q_0$ only the $F_\perp \equiv F_x + iF_y$ correlations exhibit scale invariant growth with a $t/\log t$ growth law (see Chap. 4). This raises the interesting question of how the scaling of correlations of $F_z$ and $F_\perp$ change as $|q| \to 0$, pertinent to spinor experiments which are able to resolve $q$ to uncertainties of $\delta q \lesssim 10^{-2}q_0$ \cite{223, 348}.

We study finite $q$ effects using the correlation functions

$$G_\mu(r,t) = \frac{1}{n_0^2} \langle F_\mu^*(r)F_\mu(0) \rangle_t,$$  \hspace{1cm} (5.6)

where $\mu = z, \perp$, and extract domain sizes $L_\mu$ for different times by finding the shortest distance at which $G_\mu(r,t) = 0.2G_\mu(0,t)$. The insets to Fig. 5.4 show plots of the domain sizes $L_\mu$ versus time for quenches to a range of small $q$ values. Initially the domains grow as $t^{1/2}$, like for $q = 0$. As the average domain size increases we observe that the system transitions to a new dynamic universality class: for $q < 0$ [Inset (i)] $F_\perp$ correlations stop growing and then begin to shrink, while the easy-axis order ($F_z$) transitions to growing as $t^{2/3}$; for $q > 0$ [Inset (iii)] $F_z$ correlations stop growing and then begin to shrink, while the easy-plane order ($F_\perp$) transitions to growing as $t/\log t$. Phase ordering with a growth law that depends on length scale also occurs in classical binary fluids \cite{37, 376}, however what we observe here differs in that the order parameter also changes during the transition.
Figure 5.4: The domain size $L_q$ at which coarsening transitions from isotropic to anisotropic ordering, for $q > 0$ (magenta circles) and $q < 0$ (green squares). We take $L_q = \max_t L_z$ ($L_q = \max_t L_\perp$) for $q > 0$ ($q < 0$). The line shows $L_c$ (see text). Insets show the growth of $F_\perp$ (filled circles) and $F_z$ (unfilled diamonds) domains versus $t$ for (i) $q < 0$ and (ii) $q > 0$ (as labelled in insets).
To quantify the transition from isotropic to anisotropic coarsening we identify the transition length $L_q$ as the maximum domain size for the non-ordering spin component. That is, $L_q$ is the maximum value of $L_z$ ($L_\perp$) for $q > 0$ ($q < 0$). The transition length $L_q$ is plotted in Fig. 5.4 where we observe that for positive and negative $q$ values $L_q$ is well described by a critical length $L_c = 2\hbar/\sqrt{M|q|}$. The scaling $L_q \sim L_c$ can be understood as follows. The average kinetic energy per particle arising from spatial variation of the order parameter scales as $\hbar^2/(ML_D^2)$, where $L_D$ is the average size of the growing domains. Isotropic spin rotations will be possible when this kinetic energy is larger than $|q|$, the energy required to rotate an atom’s spin away from its ground state direction. For smaller kinetic energies, such spin rotations will be suppressed and the spin direction will be confined to its ground state manifold. The transition from isotropic to anisotropic ordering is therefore expected to occur when $\hbar^2/(ML_D^2) \sim |q|$ with $L_D = L_q$. This gives the observed scaling $L_q \sim L_c$.

5.4 Conclusion

In this chapter we have explored the magnetic and gauge phase ordering dynamics of a ferromagnetic spin-1 condensate quenched to the isotropic phase. We have found that the growth of domains is scale invariant with a dynamic critical exponent of $z = 2$. We identify $\mathbb{Z}_2$ vortices that annihilate as order grows. We also observe a dynamic phase transition from isotropic to anisotropic magnetic ordering for quenches to experimentally achievable small values of $|q|$. It would be interesting to consider this dynamic transition from the perspective of non-thermal fixed points, which have been argued to be more general than equilibrium critical points [33–35]. Another interesting future direction arising from this work is to compare the SO(3) ordering observed here with ordering for an $S^3$ order parameter, which does not support topologically protected $\mathbb{Z}_2$ vortices. This ground state manifold occurs at the miscibility transition in a binary condensate [405]. It would also be interesting to explore if the phase ordering observed here progresses to a state with quasi-long range order or to a state where order decays exponentially [267]. This could shed light on the existence of a topological phase transition between bound and unbound $\mathbb{Z}_2$ vortices. We note that while ground state properties of $\mathbb{Z}_2$ vortices have been explored [261, 265, 398], we are not aware of studies of $\mathbb{Z}_2$ vortex dynamics. Such a study may be possible using variational Lagrangian methods employed for polar-core and half-quantum spin vortices [271, 302] (see Chap. 7).
Chapter 6

Thermalisation in an easy-plane ferromagnetic superfluid

In Chap. 4 it was shown that the phase ordering of a ferromagnetic spin-1 condensate in the easy-plane phase evolves in a scale invariant way, driven by the annihilation of spin vortices. In this chapter we explore the very long time phase ordering dynamics in this phase. We find that even after all vortices have annihilated the system still has not ordered. Instead, we identify an extremely slow turbulent cascade of energy, which separates out of equilibrium long wavelength spin waves from a short wavelength thermal field. This results in correlations of the order parameter decaying more rapidly than expected in equilibrium. As time progresses, the shape of the energy cascade is preserved but the cascade moves to longer wavelengths as $t^{1/4}$. We compare our results to simulations of a condensate coupled to a reservoir of energy and particles. In contrast to the microcanonical system, we find that this grand canonical system has reached an ordered state once all vortices have annihilated.

6.1 Introduction

The theory of phase ordering dynamics describes the ordering of a system following a quench across a continuous phase transition to a symmetry breaking phase. A quench to a symmetry breaking phase generates topological defects [20, 22], which are stable excitations that can only be removed through collisions with oppositely charged defects. In the conventional theory of phase ordering dynamics [37] the approach to equilibrium is driven by the annihilation of topological defects, which shows scale invariant properties at large length scales (see Sec. 2.6 for a thorough discussion).
In this chapter we explore the very long time evolution of spin order in the easy-plane ferromagnetic phase, following a quench from the polar phase (see Fig. 6.1). The easy-plane phase supports polar-core spin vortices, which consist of a circulation of transverse spin angle \([223, 268, 406]\). In Chap. 4 it was shown that order in this phase is approached via annihilation of such vortices, with the density of vortices decaying as \((t / \log t)^{-2}\). In this chapter we extend this result, and explore the system dynamics after all vortices have annihilated. We find that after all vortices have annihilated the system is still not in an ordered state under conservative evolution. Instead, we identify persistent out of equilibrium energy in long wavelength spin waves, which leads to reduced correlations. This energy moves to higher thermalised wavenumbers via an extremely slow turbulent cascade, and as a result the cascade moves to longer wavelengths. The movement of the cascade to longer wavelengths occurs in a scale invariant way, with a length scale that grows as \(t^{1/4}\), much slower than the \(t / \log t\) growth of intervortex spacing. We find that the spectrum of the cascade agrees with an analytic model of weak wave turbulence for early times, but at late times becomes steeper. Under stochastic evolution, which models condensate evolution in contact with a reservoir of energy and particles, the slow energy cascade is not present and thermalisation occurs on the time scale of vortex annihilation.

The outline of this chapter is as follows. In Sec. 6.2 we discuss the expected equilibrium properties of the ground state manifold of the easy-plane phase. Our results are presented in Sec. 6.3, where we show that following a quench to the easy-plane phase, order parameter correlations persist out of equilibrium for very long times after all topological defects have annihilated. We use ideas from weak wave turbulence to describe this observation of slow thermalisation. We find that the dynamics of the turbulent cascade is scale invariant, and from this identify the rate at which energy is thermalised. We compare these microcanonical results to results obtained from grand canonical (i.e. reservoir coupled) simulations, and find that in the grand canonical case thermalisation of the order parameter is driven by vortex dynamics. We conclude in Sec. 6.4.

### 6.2 System

Conservative dynamics of the spin-1 condensate is described by the three coupled Gross-Pitaevskii equations (GPEs) (see Sec. 3.5),

\[
i \hbar \frac{\partial \psi_m}{\partial t} = \left( -\frac{\hbar^2 \nabla^2}{2M} + m^2 q + g_m n \right) \psi_m + g_s \sum_{m'} \mathbf{F} \cdot \mathbf{f}_{mm'} \psi_{m'} \equiv \mathcal{L}_m \psi_m, \quad (6.1)
\]
Figure 6.1: Phase diagram of a ferromagnetic spin-1 condensate, showing the polar (unmagnetised) and easy-plane phases, as a function of the quadratic Zeeman energy \( q \). The ground state spin symmetry of the easy-plane phase is shown in the respective spin sphere. A quantum critical point at \( q = q_0 \equiv 2|g_s|n_0 \) separates the two phases, where \( n_0 \) is the mean condensate density. In this work we explore dynamics following a quench of \( q \) from the polar to the easy-plane phase (red arrow).

where we introduce the nonlinear evolution operators \( L_m \) for later convenience. Any arbitrary spin-1 state (ferromagnetic or polar) can be parameterized as [296],

\[
\psi = \sqrt{n} e^{i\alpha} \begin{pmatrix} \sin \beta \cos \phi e^{-i\theta} \\ \cos \beta e^{i\gamma} \\ \sin \beta \sin \phi e^{i\theta} \end{pmatrix}. \tag{6.2}
\]

In the ground state of the easy-plane ferromagnetic phase (see Fig. 6.1), \( \gamma = 0, \alpha = \phi = \pi/4 \) and \( \beta \) and \( \theta \) are uniform. The angle \( \theta \) determines the direction of transverse spin,

\[
F_\perp = n \sqrt{1 - \left( \frac{q}{q_0} \right)^2} (\cos \theta, \sin \theta, 0). \tag{6.3}
\]

The ordering of transverse spin is measured by the spatial correlation function,

\[
G(r, t) \equiv \frac{1}{n_0^2} \langle F_\perp(0) \cdot F_\perp(r) \rangle_t, \tag{6.4}
\]

where angular brackets denote an ensemble average.

Circulation of the spin angle \( \theta \) gives rise to polar-core spin vortices (PCVs). In Chap. 4 we showed that the annihilation of these defects is associated with
the phase ordering in the easy-plane phase. In addition to vortex excitations, there are three excitation branches around the ground state of (6.2): a gapless magnon branch arising from the broken easy-plane spin symmetry; a gapless phonon branch arising from the broken gauge symmetry; and a gapped magnon branch corresponding to $F_z$ fluctuations [252, 357, 358]. The phonon branch has a steep spectrum ($\sim g_n n \gg q$) while the energy gap of the gapped magnon branch is $q$. We will mainly be interested in energy scales less than $q$, in which case the system dynamics is dominated by fluctuations of $\theta$ and $\phi$. In equilibrium at low temperatures, long wavelength excitations of the spin angle $\theta$ give rise to algebraic decay of transverse spin correlations,

$$G(r) \sim r^{-\nu}. \quad \text{(6.5)}$$

To show this, one assumes quadratic fluctuations in $\theta$ and $\phi$ and calculates (6.4) via a Gaussian functional integral. In the long wavelength limit, only the $\theta$ field is important [296, 358]. The Hamiltonian then depends only on $|\nabla \theta|^2$ and is identical to the continuous limit of the XY-model. The final result for correlations at a temperature $T$ is Eq. (6.5) with $[407]$ ($k_B$ is Boltzmann’s constant),

$$\nu = \frac{M k_B T}{2 \pi \hbar^2 n_0 \sin^2 \beta}. \quad \text{(6.6)}$$

### 6.3 Results

#### 6.3.1 Transverse spin correlations

We explore the very long time phase ordering of transverse spin by numerically solving the spin GPEs (6.1) following a quench to the easy-plane phase. We solve on a 2D square grid with dimensions $l \times l = 400 \xi_s \times 400 \xi_s$ covered by an $N \times N=512 \times 512$ grid of equally spaced points, where $\xi_s \equiv h/\sqrt{2|g_s|n_0 M}$ is the spin healing length. We evolve our system using a recently developed fourth order symplectic integrator [372] to ensure that energy, particle number and $F_z$ magnetization are conserved effectively throughout the simulation. We find that the relative error accumulated across the simulation in energy and particle number is on the order of $10^{-9}$ and in $F_z$ magnetization is on the order of $10^{-5}$. In experiments in $^{87}$Rb, $g_n/|g_s| \sim 100$ [335]. We use a more modest ratio $g_n/|g_s| = 10$, which will be sufficient to suppress density fluctuations at the energy scale we are interested in. We use a time step of $0.02t_s$ for each integration step. We begin with an initial condensate in the $\psi_0$ component with a uniform density $n_0 = 10^4/\xi_s^2$. We add noise to Bogoliubov excitations on top of the initial state, which mimics quantum fluctuations and
seeds the formation of domains, see Sec. 3.11. The quadratic Zeeman energy is set at the final quench value \( q = 0.3q_0 \) for the duration of the simulation dynamics, so that the quench is effectively instantaneous at \( t = 0 \).

Figure 6.2 shows the evolution of the transverse spin correlations (6.4) with time. We invoke translational and angular symmetry to compute the ensemble average, as well as averaging over ten simulation trajectories calculated with independent initial noise. Initially, the growth of order is scale invariant and is driven by PCVs. Correlations during these times decay to zero at a length scale less than the system size. Details of this scale invariant coarsening are given in Chap. 4. We find that all vortices have decayed by a time \( t \sim 10^3t_s \). Correlations then extend to the boundary. For times \( t > 10^3t_s \) we would expect that the system will have thermalised so that the correlations decay according to the analytic prediction (6.6). We find, however, that the decay of correlations agrees with the analytic prediction (6.6) only for length scales \( r < r_a \) with \( r_a \) growing with time to \( r_a \approx 10\xi_s \) at \( t = 10^5t_s \), see Fig 6.2. Note that to calculate \( \nu \) requires knowing the system temperature. We take the temperature to be that resulting from equipartition of the energy \( \Delta E \) liberated from the quench amongst the \( 3N^2 \) numerical modes, which each behave as an oscillator with two degrees of freedom. Using \( \Delta E \) from Eq. (4.3) this gives a temperature,

\[
k_B T = \frac{q_0}{12N^2} \left( 1 - \frac{q}{q_0} \right)^2 n_0 l^2, \tag{6.7}
\]

and Eq. (6.6) gives \( \nu = 0.0113 \). For length scales \( r > r_a \), the correlations decay much more quickly than the prediction (6.6), indicating that there is excess energy at these length scales. This excess energy persists even at very long simulation times \( t = 10^5t_s \), which is two orders of magnitude longer than the time the PCVs take to annihilate.

### 6.3.2 Turbulent cascades

We can probe the cause of this slow thermalisation by analysing the energy in the spin current,

\[
\mathbf{v}_{fs} = \frac{\hbar \sin^2 \beta}{2M} \nabla \theta. \tag{6.8}
\]

This spin current is responsible for transverse spin structures, and gives rise to a current of \( F_z \) magnetization [367], see Sec. 3.9. A Helmholtz decomposition

\footnote{Note Footnote 4 in Chap. 4.}
Figure 6.2: Spatial correlations of transverse spin density for different times. Inset: correlations grow in a scale invariant way until $t \sim 500t_s$, at which point the correlation length becomes comparable to the system size. For longer simulation times the system gradually evolves toward a more ordered state. Main figure: Correlation function during the long time ordering. The dashed line shows the algebraic decay expected in equilibrium. The correlations are thermalised at small length scales over a range that grows with time. However, even after very long simulation times $t = 10^5t_s$ the correlations have not thermalised for large length scales.
resolves the Fourier transform of the vector field $\mathbf{v}_f$ into an incompressible field $\mathbf{u}_i(k)$ (with $k \cdot \mathbf{u}_i = 0$) and a compressible field $\mathbf{u}_c(k)$ (with $k \times \mathbf{u}_c = 0$), where $k$ is the wavenumber. Note the Fourier transform used is

$$\tilde{f}(k) = \frac{1}{l} \int d^2x \, f(x)e^{-ik \cdot x},$$

(6.9)

for a function $f(x)$ with Fourier transform $\tilde{f}(k)$. The compressible part gives rise to spin wave excitations; the incompressible part gives rise to PCVs [175]. An analysis of the energy spectrum of the currents $\mathbf{u}_c$ and $\mathbf{u}_i$ should then give information on how energy is distributed between spin waves and vortices, and between spin waves of different length scales.

To see how the spin current (6.8) affects the condensate energy, we substitute the wavefunction (6.2) into the mean field Hamiltonian (Eq. (3.24) with $p = 0$), allowing for fluctuations in $\theta$ and $\phi$ only (see the discussion preceding Eq. (6.5)). Fluctuations in $\theta$ can then be seen to give rise to a kinetic energy,

$$E = \frac{\hbar^2 \sin^2 \beta n_0}{2M} \int d^2x \, |\nabla \theta|^2 = \sum_k 2\pi k \left( \epsilon'_i(k) + \epsilon'_c(k) \right),$$

(6.10)

where the incompressible ($i$) and compressible ($c$) spectra are,

$$\epsilon'_\mu(k) = \frac{2Mn_0}{\sin^2 \beta} \int \frac{d\Omega_k}{2\pi} |u_\mu(k)|^2, \quad \mu = i, c.$$

(6.11)

For each spectrum we invoke rotational symmetry to average over the wavevectors $k$ with fixed amplitude. To be consistent with turbulence literature, it is useful to analyse the scaled spectras,

$$\epsilon_\mu(k) \equiv 2\pi k \epsilon'_\mu(k).$$

(6.12)

We plot the energy spectra (6.12) obtained from our numerical simulations in Fig. 6.3.\(^2\) The incompressible spectrum, Fig. 6.3(a), clearly shows a $k^{-1}$ decay when vortices are present. This power law can be analytically shown to arise from the long wavelength kinetic energy spectrum of randomly distributed vortices [175, 393]. We find that the final PCVs decay at a time $t \approx 2 \times 10^3t_s$. After this time the spectrum rapidly drops to a small residual density that grows linearly with $k$. This residual density will arise from energy equipartition of additional excitations in the system that modulate the velocity (6.8).

The compressible spectrum, Fig. 6.3(b), is of more interest to us here, as it reflects the distribution of spin wave energy and so should describe the

\(^2\)To calculate the spectra we include the spin density inside the integral in Eq. (6.10), so that the singularities in $\nabla \theta$ at the core of vortices are removed.
Figure 6.3: (a) Spectrum of the incompressible field energy $\epsilon_i$, arising from polar-core spin vortices. The $k^{-1}$ decay is expected when vortices are present. There is a rapid drop in energy after all vortices have annihilated. The small residual energy arises from additional excitations and exhibits a thermal ($k$) spectrum. (b) Spectrum of the compressible field energy $\epsilon_c$, arising from the kinetic energy of spin waves. A turbulent cascade separates low wavenumber out of equilibrium energy from high wavenumber thermalised (equipartitioned) energy. The spectrum evolves from a $k^{-1/3}$ scaling to a $k^{-4/3}$ scaling and then steepens further to roughly $k^{-3}$ and persists for a very long time.
slow thermalisation of correlations in Fig. 6.2. The nonlinear dynamics of weakly interacting waves in turbulent hydrodynamics is termed weak wave turbulence. Weak wave turbulence is characterised by the presence of power laws in the compressible energy spectrum, which arise from scale invariant cascades of energy or particles between high and low wavenumbers [408]. During the initial stages of coarsening we identify a $k^{-1/3}$ cascade in the compressible spectrum, which moves to lower and lower wavenumbers as the system orders. This is consistent with a direct cascade of energy (i.e. a flow of energy from low to high wavenumbers). As the cascade moves to wavenumbers $k < 1/\xi_s$ we observe that the spectrum evolves to a $k^{-4/3}$ scaling. In Appendix C we present a model of weak wave turbulence that predicts a $k^{-1/3}$ direct energy cascade for wavenumbers $k \xi_s > 1$ and a $k^{-4/3}$ cascade for wavenumbers $k \xi_s < 1$, with the change arising from a change in the dispersion relation of spin waves from quadratic to linear. As the cascade moves to even longer wavelengths, we observe a steepening of the cascade to roughly $k^{-3}$. This steep cascade persists for very long times $t \sim 10^5 t_s$.

We observe that the compressible spectrum evolves in a scale invariant way for times $t \gtrsim 500 t_s$ (see bottom left inset to Fig. 6.4). That is, the spectrum $\epsilon'_c$ (Eq. (6.11)) satisfies,

$$\epsilon'_c(a(t)k,t) = \epsilon'_c(k,t_0),$$

where $a(t)$ is a scaling factor satisfying $a(t_0) = 1$. We define $a(t)$ via $\epsilon'_c(a(t)^{-1},t) = 0.8 q_0$ (see bottom left inset to Fig 6.4). Figure 6.4 shows a plot of the scaled compressible spectrum $\epsilon'_c(a(t)k)$ for various times, showing good collapse onto a single function. For times $t \gtrsim 5000 t_s$, roughly corresponding to the times after all vortices have annihilated, the lengthscale $a(t)$ grows as a power law $a(t) \sim t^{1/4}$ (top right inset to Fig. 6.4). This scale invariant dynamics, which does not involve annihilation of topological defects produced from the quench, may be able to be understood within the more general framework of nonthermal fixed points [33–35]. It is worth noting that similar scaling in time has been observed for an energy cascade approaching equilibrium in the 1 + 1 dimensional $\phi^4$ model [409]. In this work energy is initially injected at low wavenumbers resulting in a wavefront that moves toward higher wavenumbers as $t^{1/4}$, leaving behind a thermalised wake. The temperature of the thermalised wake decreases as the wavefront moves forward. However, we also note that a $t^{1/4}$ scaling for a two component nonconserved order parameter (such as the complex field in a $\phi^4$ model) in one dimension is familiar from conventional phase ordering dynamics [40].

\footnote{Note that this scaling behaviour should not be confused with the earlier time scaling of longer wavelength spin waves in the analytic model of easy-plane coarsening in Sec. 4.3.4.}
Figure 6.4: Main figure: Collapse of the compressible spectrum $\epsilon'_c$ at late times onto a single curve, showing that the cascade evolves in time in a scale invariant way. Bottom left inset: Compressible spectrum before rescaling, with times shown. The $k$ value where the dashed line intersects each spectrum gives $a(t)^{-1}$. Top right inset: growth of rescaling factor $a(t)$ versus time. For times $t \gtrsim 5000 t_s$ the growth is well described by a power law $a(t) \sim t^{1/4}$. 

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6.3.3 Thermalisation of spin waves

For wavenumbers higher than those in the turbulent cascade we see a thermalisation of spin waves. Therefore the approach toward equilibrium that we have identified proceeds by a process of excess low wavenumber energy cascading to higher wavenumbers where it is able to thermalise. This effect has been observed in models of three dimensional turbulence \([410–412]\) (see also [409]). What is surprising about the cascade is how slow the energy flow is. Even for times two orders of magnitude longer than the time all PCVs take to annihilate, the spin waves have not thermalised. We find that the total energy of the compressible field \(E_c(t) = \sum_k 2\pi k c_k^c(k, t)\) is very nearly conserved after all vortices have annihilated, see Fig. 6.5. Therefore the flow of energy can be determined from the total number of excitations present in the compressible field,

\[
N_c(t) = \frac{M}{\hbar^2} \sum_k 2\pi k \frac{c_k^c(k, t)}{k^2}.
\]  

(6.14)

A plot of \(N_c\) versus time is shown in Fig 6.5 also. The decay of excitations is well described by a power law \(N_c \sim t^{-0.136}\). The small exponent of this power law is consistent with the very slow thermalisation that we have identified. Note that the exponent 0.136 is close to half the scaling exponent found in Fig. 6.4. Therefore we find empirically that \(N_c \sim a^{-1/2}\).

6.3.4 Reservoir coupled dynamics

A common idea in statistical mechanics is that a sufficiently large isolated system is expected to evolve towards the same equilibrium distribution as an open system, with conserved quantities replaced by average quantities. Here we explore differences in the approach to equilibrium of an open (grand canonical) quench to those obtained for the closed (microcanonical) quench in the previous two subsections. To model grand canonical evolution we couple our condensate to a reservoir of energy and particles with fixed temperature \(T\) and chemical potential \(\mu\). The dynamics is simulated using the simple growth stochastic Gross-Pitaevskii equations (SGPEs) [413, 414],

\[
i\hbar d\psi_m = (1 - i\kappa)(L_m - \mu)\psi_m dt + dW(r, t).
\]  

(6.15)

Here \(L_m\) is the conservative evolution operator from Eq. (6.1) and \(\kappa\) is a dimensionless damping, which we take to be 0.1. The precise value chosen for \(\kappa\) will not affect equilibrium properties, but will affect the rate that equilibrium is approached. The term \(dW(r, t)\) is Gaussian distributed complex noise with
Figure 6.5: Plot of energy (blue dots, left axis) and number of excitations (red diamonds, right axis) in the compressible field $\epsilon_c$. After a time $t \sim 10^3 t_s$, which corresponds to the time when all vortices have annihilated, the energy in the compressible field reaches a relatively steady value. The number of excitations in the field, however, continues to decay, indicating a flow of energy from low to high wavenumbers (since the total energy is relatively steady). The decay of excitations is reasonably well described by a very gradual power law $N_c \sim t^{-0.136}$. This gradual power law is consistent with a very slow thermalisation of spin waves.
\[ \langle dW^*(r,t)dW(r',t) \rangle = \frac{2\kappa k_B T}{\hbar} \delta(r - r')dt. \]  

(6.16)

The SGPEs (6.15) take the form of Langevin equations. We choose \( \mu \) and \( T \) so that the average energy and particle number of the SGPE simulations in the long time limit match the fixed energy and particle number of the GPE simulations (6.1). We fix the temperature at (6.7), which gives a final energy within 0.1\% of the total energy of the conservative results in Sec. 6.3.1. We tune the chemical potential to give a total number of particles within 0.1\% of that used in the GPE simulation; we find \( \mu = 1.0387(g_n + g_s)n_0 \) achieves this for a quench to 0.3\( q_0 \). We simulate Eqs. (6.15) using an adaptive step Runge-Kutta method, with parameters and initial condition the same as in Sec. 6.3.1.

During vortex driven phase ordering dynamics, both the microcanonical and grand canonical dynamics show comparable results for order parameter correlations (6.4), see the inset to Fig. 6.6 (c.f. Fig. 6.2). At a time \( t \approx 2500t_s \), there are only two vortices present in the grand canonical dynamics, which remain until a time \( t \approx 9700t_s \).

Once all vortices have annihilated there is a stark difference between the microcanonical and grand canonical dynamics. We have already seen in the microcanonical regime a persistent turbulent cascade that slows the rate of ordering. In the grand canonical regime we see that correlations of the order parameter reach their equilibrium value shortly after all vortices have annihilated, see Fig. 6.6. We have also carried out grand canonical simulations for quenches to \( q = 0.6q_0 \) and \( q = 0.8q_0 \), finding good agreement with Eq. (6.6) after all vortices have annihilated, see Fig. 6.6.4 We also examine the compressible component of the spin wave field, Eq. (6.12), for the quench to \( q = 0.3q_0 \), and find that the thermalisation of spin waves occurs on length scales shorter than the intervortex spacing, see Fig. 6.7. There is evidence of an initial \( k^{-1/3} \) cascade moving toward a \( k^{-4/3} \) cascade, however this dissipates as the intervortex spacing increases and so does not persist as in the microcanonical dynamics. In the grand canonical dynamics, long wavelength spin waves can rapidly thermalise with the thermal reservoir, rather than cascading to short wavelengths. It is worth pointing out that qualitative differences in microcanonical and grand canonical evolution can also be observed from images of the growing domains, see Fig. 6.8. In the

4The temperature for these additional quenches is modified according to Eq. (6.7). The chemical potential is kept fixed at \( \mu = 1.0387(g_n + g_s)n_0 \), which does lead to variable final particle number, but only by a few percent. We account for this when evaluating Eq. (6.6).
Figure 6.6: Spatial correlations of transverse spin density obtained from (single trajectory) grand canonical dynamics. Inset: Correlations for different times, showing comparable results to Fig. 6.2 for $t < 1000\tau_s$. All vortices have annihilated by a time $t = 9700\tau_s$. For times $t > 10^4\tau_s$, the correlations agree well with the equilibrium result (6.6) (dashed line). Main figure: Algebraic decay of correlations for quenches to various values of $0 < q < q_0$ (labelled in figure) at time $t = 3 \times 10^4\tau_s$. Correlations have been scaled by $G(0)$. Dashed lines show the corresponding analytic result (6.6).
Figure 6.7: Compressible energy spectrum (Eq. (6.12)) for grand canonical dynamics following a quench to $q = 0.3q_0$. For early times there is evidence of a $k^{-1/3}$ cascade evolving toward a $k^{-4/3}$ cascade. Spin waves thermalise for wave lengths shorter than the average intervortex spacing $l_v$. For the eight spectrums displayed, the length scale $2\pi/l_v$ is shown by a vertical dashed line that moves to lower wavenumbers as time progresses. The thermalisation of spin waves therefore follows the annihilation of vortices. Results averaged over ten simulation trajectories.
Figure 6.8: Evolution of the direction (i.e. spin angle $\theta$) of transverse spin domains for microcanonical (top) and grand canonical (bottom) simulations. Grid coordinates for all times are the same as in (i). The vortex count in each figure is (i) 30; (ii) 4; (iii) 0. Clearly evident in the microcanonical evolution are spin structures smaller than the intervortex spacing that are not present in the grand canonical evolution. These structures become apparent at early times in the microcanonical evolution, and persist for very long times.
microcanonical dynamics we see spin structures on length scales shorter than the intervortex spacing that are not present in the grand canonical dynamics.

6.4 Conclusion

In this chapter we have explored the very long time phase ordering of an easy-plane ferromagenetic superfluid. For microcanonical dynamics, we find that once all vortices have annihilated there persists a slow spin wave turbulent cascade, which moves to lower wavenumber in a scale invariant way as time progresses. It would be interesting to see if this spin wave cascade relates to the fractal scaling of the structure factor in Fig. 4.7(b). In grand canonical dynamics, we find that a spin wave turbulent cascade does not persist for length scales shorter than the intervortex spacing, and ordering proceeds via vortex annihilation. This work highlights the role that turbulence can play in conservative evolution and offers a new perspective on the theory of phase ordering dynamics for conservative evolution. It is hoped that the physics we observe is not unique to spin-1 condensates. Note that the fluctuations that we deem to be important in our analysis are density-phase fluctuations, and these also appear in scalar condensates [415].
Chapter 7

Dynamics of polar-core spin vortices

A ferromagnetic spin-1 condensate supports polar-core spin vortices (PCVs) in the easy-plane phase. In this chapter we derive a model for the dynamics of these PCVs using a variational Lagrangian approach. The PCVs behave as massive charged particles interacting under the two dimensional Coulomb interaction, with the mass arising from interaction effects within the vortex core. We compare this model to numerical simulations of the spin-1 Gross-Pitaevskii equations and find semi-quantitative agreement. In addition, the numerical results suggest that the PCV core couples to spin waves, and this affects the PCV dynamics even far from the core. We identify areas of further research that could extend the model of PCV dynamics presented here.

7.1 Introduction

Spinor Bose-Einstein condensates can exist in various ferromagnetic or antiferromagnetic phases depending on the nature of the spin dependent interactions and Zeeman shifts of the spin levels. Associated with this range of symmetry breaking phases is a rich array of topological defects [299, 335]. In a ferromagnetic spin-1 condensate, the Zeeman shifts arising from an external field can be tuned so that the condensate magnetizes in a plane perpendicular to the external field. This inplane transverse magnetization breaks the U(1) rotational symmetry about the direction of the external field. This phase is termed the easy-plane phase and can support polar-core spin vortices (PCVs). The transverse magnetization angle exhibits a phase winding around a PCV, with a filled but unmagnetized core. The easy-plane symmetry breaking and PCVs have been observed in situ experimentally in a spin-1 condensate [223].
PCVs form spontaneously during a quench to the easy-plane phase via the Kibble-Zurek mechanism [224, 225, 227], and in Chap. 4 we showed that PCVs play an integral role in the post-quench phase ordering dynamics. PCVs may also play a role in spin turbulence [274, 279] and, at non-zero temperature, could possibly give rise to a Berezinskii-Kosterlitz-Thouless transition to the easy-plane phase. Despite the relevance of PCVs to a wide variety of interesting physical processes, a comprehensive study of their properties is lacking.

A model of the dynamics of PCVs was presented by Turner in [268], where hydrodynamic arguments were used to show that the PCVs should interact like massive charged particles according to the two dimensional Coulomb’s law. The presence of a logarithmic Coulomb interaction is known in scalar vortices [322, 416], however the presence of the vortex mass makes Turner’s dynamic model of PCVs vastly different from scalar vortex dynamics. Although a hydrodynamic analysis captures much of the unique properties of PCV dynamics, accurate details of the vortex mass are not described, and this mass property arising from core effects plays an essential role in the PCV dynamics. In this chapter we give a mean field microscopic derivation of the Turner model using a variational Lagrangian approach. This method provides a systematic way to explore the properties of the PCV mass term. We test the Turner model using numerical simulations of the spin-1 Gross-Pitaevskii equations and find semi-quantitative agreement. In addition we find numerically that the coupling of spin waves to a PCV core may have a substantial effect on the PCV dynamics. Our microscopic treatment offers obvious paths for future investigation that could lead to refinements of the Turner model.

The outline of this chapter is as follows. In Sec. 7.2 we introduce the background formalism for our study of PCVs. In Sec. 7.3 we develop a model of PCV dynamics using a variational Lagrangian approach. In Sec. 7.4 we present numerical simulations of a PCV vortex antivortex pair and compare to predictions of the model, which highlights the important role spin waves may play in the PCV dynamics. We conclude in Sec. 7.5 with a summary and suggestions for future research.

7.2 Formalism

7.2.1 Ground states

By varying the quadratic Zeeman energy, the ground state of a ferromagnetic spin-1 condensate can be varied between different magnetic states, Fig. 7.1
Figure 7.1: Ground state magnetic phases for different values of the quadratic Zeeman energy $q$ for an external field along $z$. The direction of the arrows within each sphere signify the ground state magnetization directions. The polar phase has no magnetization. We choose $F_z$ to be parallel to $z$. PCVs can form as defects in the easy-plane phase (b).

(see Sec. 3.7 for a thorough discussion). For $0 < q < q_0$, the system magnetizes in a plane perpendicular to the applied field, termed the easy-plane phase. The ground state wave function of this phase is

$$\psi_g = \sqrt{n_0} e^{i\alpha} \begin{pmatrix} \sin \beta \sqrt{2} e^{-i\theta} \\ \cos \beta \sqrt{2} e^{-i\theta} \end{pmatrix}, \quad (7.1)$$

where $n_0$ is the mean density, $\cos(2\beta) = q/q_0$, and $\alpha$ and $\theta$ are phases arising from the gauge and transverse spin symmetries respectively. This wave function gives rise to a spin density

$$F = n_0 \sqrt{1 - \left(\frac{q}{q_0}\right)^2} (\cos \theta, \sin \theta, 0), \quad (7.2)$$

so that $\theta$ gives the transverse spin angle.

### 7.2.2 Polar-core spin vortices

Phase winding of the transverse spin angle $\theta$ gives rise to PCVs. This corresponds to a circulation of the superfluid flow of $F_z$ magnetization, since the $F_z$ superfluid current is proportional to $\nabla \theta$ [367]. The core of a PCV is filled by the $m = 0$ component. The core is therefore in the polar phase and removes the phase singularity in the transverse magnetization.
A PCV is a composite vortex consisting of vortices of opposite charge in the $m = \pm 1$ components, see Eq. (7.1) and Fig. 7.2. There is a mass flow of $\psi_1$ around the vortex in the $m = 1$ component and an equal but opposite flow of $\psi_{-1}$ around the vortex in the $m = -1$ component. This gives rise to a net flow of $F_z$ magnetization, but no net mass flow.

The state of a single PCV with the centre of circulation of the $\psi_{\pm 1}$ components at the origin can be written as

$$
\psi(x) = \sqrt{n_0} \begin{pmatrix} g_1(x) \sin \beta e^{-i\theta_1} \\ g_0(x) \cos \beta \\ g_{-1}(x) \sin \beta e^{i\theta_{-1}} \end{pmatrix},
$$

(7.3)
where $\kappa \theta_m$ are phase profiles that give rise to a $2\pi\kappa$ phase winding of the $m$th spin component about the origin and $\kappa \in \mathbb{Z} \setminus \{0\}$ gives the charge of the total PCV. The phase winding of the transverse spin is anticlockwise around a positively charged PCV and clockwise around a negatively charged PCV.

Far from the vortex core the vortex energy density is simply the kinetic energy arising from the phase winding. This will be minimised with a circular phase profile,

$$\theta_m = \text{phase}(x + iy),$$

where phase($z$) returns the phase angle of the complex argument $z$. Within the vortex core the phase profile can be modified due to the spin exchange energy, as will be discussed later. To avoid a kinetic energy singularity at the vortex centres, we include (unspecified) real core structures $g_m \geq 0$ in each of the spin components. These core functions should satisfy,

$$g_{\pm 1}(0) = 0,$$

$$g_0(0) \cos \beta \approx 1,$$

$$g_m(x) \approx 1 \text{ for } |x| > \xi_s.$$  \hspace{1cm} (7.5)

We will not investigate the core structure precisely in this work. However, we will discuss implications of the core structure. We consider the case of no net $F_z$ magnetization,

$$\int d^2x \, F_z = 0.$$  \hspace{1cm} (7.6)

From symmetry, the core functions of the single PCV (7.3) will therefore also satisfy $g_1(x) = g_{-1}(x)$.

With multiple PCVs, we generalise Eq. (7.3) to [175, 269, 391, 392]

$$\psi(x) = \sqrt{n_0} \begin{pmatrix} \frac{\sin \beta}{\sqrt{2}} \prod_k g_{1k}(x) e^{-i\kappa_k \theta_{1k}} \\ \cos \beta \prod_k g_{0k}(x) \\ \frac{\sin \beta}{\sqrt{2}} \prod_k g_{-1k}(x) e^{i\kappa_k \theta_{-1k}} \end{pmatrix},$$

where a subscript $k$ denotes the $k$th PCV. Now the $\kappa_k \theta_{mk}$ are phase profiles that give rise to a $2\pi\kappa_k$ phase winding of the $m$th spin component about the point $x_{mk} \equiv (X_{mk}, Y_{mk})$. The points $x_{mk}$ are centres of circulation, see Fig. 7.2.
7.2.3 Lagrangian formulation

The fields $i\psi_m(x, t)$ and $\psi_m^*(x, t)$ are conjugate fields satisfying Hamilton’s equation of motion [335]

$$i\hbar \frac{\partial \psi_m(x, t)}{\partial t} = \frac{\delta H}{\delta \psi_m^*(x, t)}.$$  \hspace{1cm} (7.8)

We obtain the Lagrangian by means of a Legendre transform of the spin-1 Hamiltonian,

$$L = i\hbar \sum_m \int d^2x \frac{\partial \psi_m(x, t)}{\partial t} \psi_m^*(x, t) - H,$$  \hspace{1cm} (7.9)

where $H$ is the Hamiltonian (3.24) with $p = 0$. We decompose the Lagrangian as follows,

$$L = L_{\text{int}} + \sum_m L_m,$$  \hspace{1cm} (7.10)

where

$$L_m = \int d^2x \psi_m^* \left[ i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2 \nabla^2}{2M} \right] \psi_m,$$  \hspace{1cm} (7.11)

and

$$L_{\text{int}} = -\int d^2x \left[ \frac{g_n}{2} n^2 + \frac{g_s}{2} |\mathbf{F}|^2 + q \left( |\psi_1|^2 + |\psi_{-1}|^2 \right) \right].$$  \hspace{1cm} (7.12)

7.3 A model of vortex dynamics

7.3.1 The variational Lagrangian

The dynamics of PCVs governed by the GPEs are complicated by the possibility of dynamic core effects and coupling between PCVs and other excitations such as spin waves. These effects are ignored in Turner’s model, and we will do the same here. We use a variational approach using the wavefunction ansatz (7.7) with the vortex positions $\mathbf{x}_{mk}(t) = (X_{mk}(t), Y_{mk}(t))$ as variational parameters. We assume static core structures

$$g_{mk}(\mathbf{x}) = g_m \left( \mathbf{x} - \mathbf{x}_{mk}(t) \right),$$  \hspace{1cm} (7.13)

and static phase profiles. Assuming static core structures and phase profiles neglects possibly interesting core dynamics. For example, we will find that
coupling to spin waves may have an important effect on the PCV dynamics. However, essential parts of the PCV dynamics are still captured within the approximations made here.

We substitute (7.7) into the Lagrangian (7.9) and minimise the action \( S = \int dt L \) with respect to the variational parameters \((X_{mk}, Y_{mk})\). This is equivalent to assuming that the variational parameters obey Lagrange’s equations of motion.

The terms \( L_m (7.11) \) will be dominated by phase variation outside the vortex cores. We therefore ignore the core structures in the evaluation of these terms and assume circular phase profiles (7.4). The \( L_m \) terms are then identical to the Lagrangian for three non-interacting scalar condensates and, after appropriate regularization, become (see e.g. [392])

\[
L_m = -\frac{\sigma M}{2} \sum_k m \kappa_k \left( \dot{X}_{mk} Y_{mk} - \dot{Y}_{mk} X_{mk} \right) + \frac{\sigma}{2} \sum_{j,k>j} \kappa_j \kappa_k \ln \left| \frac{X_{mk} - X_{mj}}{l} \right|^2, \tag{7.14}
\]

where

\[
\sigma = \frac{\pi \hbar^2 n_0 \sin^2 \beta}{2M}, \tag{7.15}
\]

and \( l \) is the system size.

The term \( L_{\text{int}} (7.12) \) can be written as

\[
L_{\text{int}} = L_D + L_{SE}. \tag{7.16}
\]

The density term \( L_D \) is

\[
L_D = (g_s - g_n) \int d^2x |\psi_1|^2 |\psi_{-1}|^2 - (g_s + g_n) \int d^2x |\psi_0|^2 \left( |\psi_1|^2 + |\psi_{-1}|^2 \right). \tag{7.17}
\]

The terms in \( L_D \) depend only on the core structures \( g_m \) and will be constant for \( g_m(x) = 1 \). We have ignored \( |\psi_m|^2 \) and \( |\psi_m|^4 \) terms in (7.17), as these only contribute a constant term for static core structures \( g_m \) and nonoverlapping PCV cores. The remaining \textit{spin exchange} term is

\[
L_{SE} = -2g_s \text{Re} \int d^2x \psi_1^* \psi_{-1} \psi_0^2, \]

\[
= -g_s n_0^2 \sin^2 \beta \cos^2 \beta \times \int d^2x \prod_k g_{1k} g_{-1k} g_{0k}^2 \cos (\kappa_k \theta_{1k} - \kappa_k \theta_{-1k}). \tag{7.18}
\]
This spin exchange interaction depends not only on the core structures $g_m$ but also on the phase profiles $\theta_{\pm 1k}$. This term will therefore affect the dynamics even with negligible core structures, i.e. with $g_m(x) = 1$. The spin exchange interaction allows for two $m = 0$ atoms to scatter into $m = \pm 1$ atoms, and vice versa. This interaction is not present in a multi-component condensate of different atoms.

We are now in a position where we can write down equations of motion for the variational parameters $(X_{mk}, Y_{mk})$. We do this implicitly by introducing “centre of mass” and relative coordinates for each PCV (see Fig. 7.2),

\[ R_k = \frac{x_{1k} + x_{-1k}}{2}, \]
\[ r_k = x_{1k} - x_{-1k}. \]

We assume dynamics where the spacing between vortex $k$ and the remaining PCVs is much larger than the relative coordinate $r_k$, i.e. $r_k \ll |R_k - R_j|$. We can therefore set $|x_{1k} - x_{-1j}| \approx |x_{-1k} - x_{-1j}| \approx |R_k - R_j|$ and talk about a single PCV being at position $R_k$.

This gives the full Lagrangian

\[ L = \frac{\sigma}{2} \left[ -\frac{M}{\hbar} \sum_k \kappa_k \left( \dot{X}_k \dot{y}_k - \dot{y}_k X_k - \dot{Y}_k x_k + \dot{x}_k Y_k \right) \right. \]
\[ \left. + 2 \sum_{j,k \neq j} \kappa_j \kappa_k \ln \left| \frac{R_k - R_j}{l} \right|^2 + f_D + f_{SE} \right], \]

where

\[ f_D \equiv \frac{2}{\sigma} L_D, \]

and

\[ f_{SE} \equiv \frac{2 \cos^2 \beta}{\pi \xi_s^2} \int \frac{d^2 x}{\prod_k g_{1k} g_{-1k} g_{0k}^2} \cos (\kappa_k \theta_{1k} - \kappa_k \theta_{-1k}). \]

Lagrange’s equations of motion are

\[ \frac{\partial L}{\partial \chi_k} = \frac{d}{dt} \frac{\partial L}{\partial \dot{\chi}_k}, \]

for the variational parameters $\chi_k \in \{X_k, Y_k, x_k, y_k\}$. The momentum conjugate to a spatial coordinate $\chi$ is $p_\chi \equiv \partial L/\partial \dot{\chi}$. The Lagrangian (7.21) therefore
gives,

\[ p_{X_k} = -\frac{M\sigma}{2\hbar} \kappa_k y_k, \quad (7.25a) \]
\[ p_{Y_k} = \frac{M\sigma}{2\hbar} \kappa_k x_k, \quad (7.25b) \]
\[ p_{x_k} = -\frac{M\sigma}{2\hbar} \kappa_k Y_k, \quad (7.25c) \]
\[ p_{y_k} = \frac{M\sigma}{2\hbar} \kappa_k X_k. \quad (7.25d) \]

The conjugate variable relations (7.25) show that a centre of mass co-
ordinate \( R_k \) is conjugate to the rotated relative coordinate \( r_k \times \hat{\mathbf{z}} \). This differs from the scalar vortex case, where the \( X \) and \( Y \) positions of a single vortex are conjugate pairs [417]. The four Lagrange’s equations of motion that describe the PCV dynamics are

\[ \dot{X}_k = \frac{\hbar}{2M\kappa_k} \frac{\partial (f_D + f_{SE})}{\partial y_k}, \quad (7.26a) \]
\[ \dot{Y}_k = -\frac{\hbar}{2M\kappa_k} \frac{\partial (f_D + f_{SE})}{\partial x_k}, \quad (7.26b) \]
\[ \dot{y}_k = -\frac{\hbar}{M} \left[ 2 \sum_{j \neq k} \kappa_j \frac{X_k - X_j}{|R_k - R_j|^2} + \frac{1}{2\kappa_k} \frac{\partial (f_D + f_{SE})}{\partial X_k} \right], \quad (7.26c) \]
\[ \dot{x}_k = \frac{\hbar}{M} \left[ 2 \sum_{j \neq k} \kappa_j \frac{Y_k - Y_j}{|R_k - R_j|^2} + \frac{1}{2\kappa_k} \frac{\partial (f_D + f_{SE})}{\partial Y_k} \right]. \quad (7.26d) \]

7.3.2 Conservation laws

Lagrange’s equations of motion (7.26) are invariant under translations in
space, translations in time and rotations. Invoking Noether’s theorem, these
invariances give rise to the following conserved quantities. Invariance under
translations in time gives rise to conservation of the Hamiltonian

\[ H = -\frac{\sigma}{2} \left[ 2 \sum_{j \neq k} \kappa_j \kappa_k \ln \left| \frac{R_k - R_j}{l} \right|^2 + f_D + f_{SE} \right]. \quad (7.27) \]

Invariance under spatial translations gives rise to conservation of the total
linear momentum

\[ \mathbf{P} = \sum_k \langle p_{X_k}, p_{Y_k} \rangle = \frac{M\sigma}{2\hbar} \sum_k \kappa_k \hat{\mathbf{z}} \times \mathbf{r}_k. \quad (7.28) \]
Invariance under rotations gives rise to conservation of the total angular momentum

$$L = \sum_k \left[ (X_k, Y_k, 0) \times (p_{Xk}, p_{Yk}, 0) + (x_k, y_k, 0) \times (p_{xk}, p_{yk}, 0) \right],$$

$$= \frac{M \sigma}{\hbar} \sum_k \kappa_k (r_k \cdot R_k) \hat{z}. \quad (7.29)$$

### 7.3.3 Reduction to Turner’s model of PCV dynamics

In scalar vortex dynamics, the vortex cores can be neglected to obtain a point vortex model [392]. In contrast, PCV dynamics depend crucially on the internal vortex coordinates $r_k$. Although the vortex cores of single spin levels are approximated as static, the total core structure of the PCV can still exhibit dynamics. This arises from the term $f_D + f_{SE}$ in the Lagrangian (7.21), which makes dealing with the Lagrangian (7.21) numerically complicated. Furthermore, this term depends crucially on the details of the core structures $g_m$. To make progress, we follow [268] and assume a simple form for this term (ignoring a constant shift),

$$f_D + f_{SE} = -\frac{a}{2\xi_s^2} \sum_k \kappa_k^2 r_k^2, \quad (7.30)$$

where $a$ is an empirical parameter that we will find to be on the order of 0.1. In this approximation we are assuming that the interaction energy of a single PCV core depends only on the coordinate $r_k$, and not on the details of any other PCVs. (This assumption also allows us to neglect the direction of $r_k$.) Furthermore, we are assuming the interaction energy does not deviate much from the energy minimum at $r_k = 0$, so that we consider terms up to second order in $r_k$ only. We expect the interaction to be an analytic, even function of $r_k$ so that there is no linear term in the expansion. We will term the interaction energy the “stretch energy”, as it arises from the energy cost of separating the $\psi_1$ and $\psi_{-1}$ components of a PCV, which stretches the PCV core.

If we suppose that the stretch energy can be considered quadratic for $r_k \lesssim \xi_s$, then the quadratic approximation will hold for stretch energies $\lesssim \hbar^2 n_0/M$, see Eq. (7.27). The energy required to stretch the PCV core comes from changes in the logarithmic interaction term in (7.27) prior to vortex annihilation. Therefore the stretch energy will remain on the order of $\hbar^2 n_0/M$ as long as $\ln(D_i/\xi_s)$ is not much larger than 1, for initial vortex separations $D_i$ and zero initial stretching. It is plausible that the parameter $a$ may also have an additional dependence on the magnitude of $\kappa_k$. We do
not consider this possibility here, and note that if all vortices have the same
magnitude of charge then this dependence will not impact the model apart
from a fixed change of $a$. We expect that the PCV core structure and phase
profile close to the core will depend on the quadratic Zeeman energy $q$, and
therefore $a$ will depend on $q$ also.

With the Lagrangian simplified by Eq. (7.30), we can evaluate Lagrange’s
equations of motion (7.26). The two coupled first order equations of motion
for $R_k$ and $r_k$ can be decoupled by taking a second time derivative of $R_k$.
We then obtain a single equation of motion

$$
\frac{d^2 R_k}{dt^2} = \frac{\hbar^2 a}{M^2 \xi^2} \frac{\kappa_k \kappa_j}{|R_k - R_j|^2} (R_k - R_j), \quad (7.31)
$$

$$
r_k = \frac{2M \xi^2}{\hbar a \kappa_k} \frac{dR_k}{dt} \times \hat{z}. \quad (7.32)
$$

We can write this in a form analogous to a single particle classical mechanics
model by noting that $r_k \propto |dR_k/dt|$ so that the stretch energy (ignoring a
constant shift) can be written as,

$$
H_s \equiv -\frac{\sigma}{2} (f_D + f_{SE}),
= \frac{\sigma a}{4 \xi^2} \sum_k \kappa_k^2 f_k^2,
= \frac{1}{2} m_v \sum_k \left| \frac{dR_k}{dt} \right|^2, \quad (7.33)
$$

with the vortex mass $m_v$ defined as

$$
m_v \equiv \frac{\pi M n_0 \xi^2 \sin^2 \beta}{a}. \quad (7.34)
$$

Equations (7.31), (7.32) can then be written as

$$
m_v \frac{d^2 R_k}{dt^2} = 2\sigma \sum_{j \neq k} \kappa_k \kappa_j \frac{R_k - R_j}{|R_k - R_j|^2}, \quad (7.35)
$$

$$
r_k = \frac{\hbar m_v}{M \sigma \kappa_k} \frac{dR_k}{dt} \times \hat{z}. \quad (7.36)
$$

Equation (7.35) resembles the dynamics of classical charged particles of mass
$m_v$ moving under the two dimensional Coulomb interaction

$$
U = -\sigma \sum_{j,k>j} \kappa_k \kappa_j \ln \left| \frac{R_k - R_j}{l} \right|^2. \quad (7.37)
$$

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The Coulomb interaction is the kinetic energy of the condensate arising from the phase winding of the vortices. The total energy \( H + U \) will be conserved under the dynamics of Eqs. (7.35), (7.36). This is the model proposed in [268].

The total linear momentum (7.28) can be written as

\[
P = \frac{1}{2} \sum_k m_v \frac{dR_k}{dt}.
\]

The total angular momentum (7.29) can be written as

\[
L = \sum R_k \times m_v \frac{dR_k}{dt}.
\]

### 7.4 Numerical results for a polar-core vortex antivortex pair

#### 7.4.1 GPE dynamics of a vortex antivortex pair

We consider the evolution of the following initial state,

\[
\psi = \sqrt{n_0} \begin{pmatrix} \sin \beta e^{-i(\theta_2 - \theta_1)} \\ \frac{\sqrt{2}}{\sqrt{2}} e^{i(\theta_2 - \theta_1)} \end{pmatrix},
\]

where

\[
\begin{align*}
\theta_1(x, y) &= \text{phase} \left( x + \frac{D}{2} + iy \right), \\
\theta_2(x, y) &= \text{phase} \left( x - \frac{D}{2} + iy \right),
\end{align*}
\]

gives rise to two oppositely charged PCVs with centres of circulation at positions \( R_1 = (-D/2, 0) \) (charge \( \kappa_1 = -1 \)) and \( R_2 = (D/2, 0) \) (charge \( \kappa_2 = 1 \)), see Fig. 7.3. Although higher charged vortices are possible they will be energetically unstable, and will decay into singly charge vortices that preserve topological charge. We do not include an initial core structure in (7.40). However, an appropriate core structure will form from the GPE evolution through the interaction with other excitations, producing a small

---

\( ^1 \)We note that our “Coulomb constant” is larger than that in [268] by a factor of \( \pi \). However, an expression for the PCV mass is not given in [268], and so if our PCV mass (7.34) is also larger by a factor of \( \pi \) then the relevant quantity \( \sigma/m_v \) would be the same as that in [268].
Figure 7.3: Evolution of transverse spin angle $\theta$ with time. First frame shows the full system while subsequent frames show the area within the dotted square marked in the first frame. Clearly visible are the two oppositely charged PCVs attracting, colliding and annihilating after a time $t \approx 50 t_s$. The average speed of the vortex is on the order of the characteristic spin wave speed $\xi_s / t_s$, see Eq. (7.45). After annihilation a spin pulse propagates away from the vortex collision with a speed on the order of the characteristic spin wave speed.

number of excitations on top of the vortex state. The vortices are initially unstretched, i.e. $r_1 = r_2 = 0$. Note that the initial state (7.40) has no net $F_z$ magnetization, Eq. (7.6).

We solve the spin-1 Gross-Pitaevskii equations (see Sec. 3.5) with the initial condition (7.40) with periodic boundary conditions. Further numerical details are provided in Sec. 3.11. We choose $q = 0.3q_0$, $n_0 = 10^4 / \xi^2_s$ and $g_n / |g_s| = 10$. We use a $4096 \times 4096$ grid with side lengths $l = 200 \xi_s$ and initial vortex separation $D = 20 \xi_s$. With this initial condition there is a discontinuity in the phase profile along the horizontal boundaries. This gives rise to a non-zero kinetic energy at these boundaries, which will manifest itself as spin waves and interact with the vortices. To mitigate this effect we choose $D < l$, since far from a dipole the phase profile decays quadratically with distance. Periodic boundary conditions also introduce image charges and prevent simulating systems with a net phase winding around the boundary, for example two same charged PCVs.

Figure 7.3 shows the evolution of the angle of the transverse spin. As predicted by Eq. (7.35), the oppositely charged PCVs attract.\(^2\) The vortices

\(^2\)Note that the PCV dynamics is very different from scalar vortex dynamics. In the
annihilate at time $t \approx 50t_s$ and their energy is liberated as a propagating spin pulse (frames with $t > 50t_s$). The pulse travels at a speed on the order of the characteristic spin wave speed $\xi_s/t_s$ [358]. Associated with this collision is a stretching of the PCVs perpendicular to their centre of mass velocity, as predicted by Eq. (7.36). We show this stretching in Fig. 7.4 by plotting the $F_z$ magnetization

$$F_z = |\psi_1|^2 - |\psi_{-1}|^2.$$  \hfill (7.42)

A separation of the positive $F_z$ peak from the negative $F_z$ dip within each PCV core arises from the separation of the cores of the $\psi_{\pm 1}$ vortices. Included in this figure are white (black) crosses that mark the centre of circulation for vortices in the $\psi_1$ ($\psi_{-1}$) components i.e. the $x_{1k}$ ($x_{-1k}$) points. The direction of stretching is consistent with Eq. (7.36).

The initial state (7.40) gives rise to PCVs with centre of mass positions $R_1 = (-D/2, 0)$ (charge $\kappa_1 = -1$) and $R_2 = (D/2, 0)$ (charge $\kappa_2 = 1$), both with zero initial stretching, $r_1 = r_2 = 0$. With this initial condition, Eqs. (7.35), (7.36) can be solved analytically to obtain

$$D(t) = D(0) \exp \left[ - \left( \operatorname{erf}^{-1} \left( \frac{t}{t_{\text{coll}}} \right) \right)^2 \right],$$ \hfill (7.43)

$$r(t) = \frac{\hbar}{M} \sqrt{\frac{2m_e}{\sigma}} \operatorname{erf}^{-1} \left( \frac{t}{t_{\text{coll}}} \right).$$ \hfill (7.44)

Here

$$t_{\text{coll}} \equiv D(0) \sqrt{\frac{\pi m_e}{8\sigma}} \sim \frac{t_s}{\xi_s} D(0),$$ \hfill (7.45)

gives the time the vortices take to collide. Note that this solution predicts that the average speed of the vortex should be independent of the initial separation and should be on the order of the characteristic spin wave speed $\xi_s/t_s$. The image charges that are introduced in the numerics from the periodic boundary conditions could be accounted for when solving Eqs. (7.35) and (7.36) [420]. For the small ratios of $D/l$ in Fig. 7.3, the image charges will have only a small effect on the dynamics and so we ignore them here.

3To see where this solution comes from, note again that the equation of motion for $D$ resembles that of a classical particle moving in a conservative potential. The equation of energy conservation can be rearranged to obtain an equation of motion for $\dot{D}$, which can be solved by integration to obtain $t$ as a function of $D$ [419]. Inverting gives $D$ as a function of $t$. 

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Figure 7.4: Plot of $F_z$ magnetization, Eq. (7.42), for various times during the PCV collision. For frames with $t < 50t_s$, the peak and dip in $F_z$ arise from the separation of the cores of the $\psi_{\pm 1}$ vortices. This indicates that each PCV stretches in a direction perpendicular to its centre of mass motion as the PCVs attract. The centre of circulation for vortices in the $\psi_1$ ($\psi_{-1}$) components, i.e. $x_{1k}$ ($x_{-1k}$), are marked by white (black) crosses. The direction of stretching is consistent with Eq. (7.36).
Figure 7.5: (a) PCV separation versus time. Blue dots are numerical data, solid black line is the analytic prediction (7.43) with $m_v$ as a fitting parameter. (b) Separation of the $\psi_{\pm 1}$ vortex centres versus time for a single PCV. Blue dots are numerical data, solid black line is the analytic prediction (7.44) using $m_v$ obtained from the fit in (a). For a short time $t \lesssim 3t_s$ during the initial evolution, the numerical result in (b) lies slightly above the analytic prediction. After this, the numerical result lies below the analytic prediction. Also visible in the numerical results in (a) and (b) is an oscillation at a frequency on the order of the characteristic spin wave frequency, $\omega_s \equiv 1/t_s$, which is not predicted by the analytic model (7.43), (7.44). We suspect the oscillation arises from coupling to spin waves. As a PCV stretches, interactions within the core may couple the core to other spin excitations so that energy is lost in the form of spin waves, thus reducing the stretching below the analytic prediction.
Fig. 7.5(a) shows the numerical result for the PCV separation $D$ against time. Also included in Fig. 7.5(a) is the analytic solution (7.43) with the vortex mass as a fitting parameter. From this fit we obtain that $a \approx 0.135$ so that the vortex mass comes out as $m_v \sim 10 M n_0^2 \xi^2$, which is roughly an order of magnitude larger than the physical mass of the vortex core. We have carried out similar simulations for interaction ratios $g_n/|g_s| = 2.5$ and $g_n/|g_s| = 40$, which also gave $a \approx 0.135$. We find that fluctuations of total density outside the PCV cores are essentially zero and within the PCV cores are small, since the relevant energy scale of our system, $|g_s| n_0$, is much less than the energy scale of density fluctuations, $g_n n_0$. In experiments with $^{87}$Rb, $g_n/|g_s| \sim 100$ [335] so that density fluctuations will be even smaller.

Fig. 7.5(b) shows the numerical result for the vortex stretching against time, along with the analytic prediction (7.44) using the fitted mass from (a). For a short time $t \lesssim 3t_s$ during the initial evolution, the numerical result lies slightly above the analytic prediction. After this, the numerical result lies below the analytic prediction. Also visible in the numerical results (a) and (b) is an oscillation with a period on the order of $2 \pi t_s$. We suspect that these oscillations arise from coupling to spin waves, a possibility that was also suggested in [268]. As a PCV stretches, it is possible that interaction effects inside the PCV core couple the core to other spin excitations, allowing the stretch energy to leave the core as spin waves. The analytic prediction in Fig. 7.5(b) would then be expected to be larger than the numerical result since energy lost to other excitations is not accounted for in the analytic model (7.35), (7.36) but is in the GPE dynamics. The possible spin waves include transverse and axial spin excitations. Additional core dynamics may also be present that do not result in the emission of spin waves, for example spin oscillations that change the core structure with time while preserving the core energy. Note that the time scale of vortex dynamics is comparable to the time scale of the spin interaction energy, so that PCV motion will likely not be adiabatic compared to core dynamics arising from the spin interaction.

The analytic results in Figs. 7.5(a) and (b) capture the main qualitative behaviours of PCVs, namely that PCVs of opposite charge attract and stretch perpendicular to their centre of mass motion as they accelerate toward each other. Therefore the analytic model captures the drastic qualitative difference between the dynamics of PCVs and scalar vortices. The analytic model overestimates the PCV stretching but does give a reasonable estimate for the

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4Periodic boundary conditions generate a discontinuity in the initial phase profile along the horizontal boundaries. As the system evolves, this discontinuity will smooth out and generate additional excitations during the vortex evolution, which may also influence the PCV dynamics. Excitations will also be produced in the initial GPE evolution as the PCV cores form.
Figure 7.6: (a) The spin exchange energy, Eq. (7.46), as a function of the PCV separation $D$ and the PCV stretching $r$. (b) Solid black line shows data from (a) for $D = 20\xi_s$, as a function of $r$. For $r \lesssim 3\xi_s$ the energy increases quadratically with $r$. A quadratic fit to the data at small $r$ is shown by blue crosses. For $r > 3\xi_s$ the energy increases linearly with $r$. A linear fit to the data at large $r$ is shown by red crosses. (c) Solid black line shows data from (a) for $r = 2\xi_s$, as a function of $D$. The energy increases logarithmically with $D$. A logarithmic fit to the data is shown by blue crosses.

...time scale of the vortex annihilation. Finding accurate vortex core ground states (e.g. see [406]) could be used to determine the interaction term (7.16) more accurately. The effect of spin waves has been neglected in the analytic model presented here. Including the effect of spin waves exactly would likely be difficult. However it may be possible to approximate this effect, or include the effect phenomenologically.
7.4.2 The spin exchange interaction energy and confinement

The interaction term \( (7.16) \) depends on both the internal phase profile of a PCV and the core structure. For \(|x| \gg \xi_s\), we have \( g_m(x) \approx 1 \) [see Eq. (7.5)] and the vortex phase profile will be close to circular. In this regime only the spin-exchange part of the interaction term \( (7.16) \) will affect the dynamics. For the case of two singly charged PCVs of opposite charge, the energy corresponding to this term takes the form

\[
H_{SE} \approx \frac{\sigma \cos^2 \beta}{\pi \xi_s^2} \int d^2x \left( 1 - \prod_{k=1,2} \cos (\theta_{1k} - \theta_{-1k}) \right),
\]

for circular phase profiles \( \theta_{\pm 1k} \), Eq. (7.4). We take the zero point energy to be the energy of two unstretched PCVs. Fig. 7.6(a) shows a plot of the spin-exchange energy \( (7.46) \) versus the vortex separation, \( D \), and vortex stretching, \( r \), obtained by numerically integrating Eq. (7.46). For each evaluation of the energy, we fix \( X_{\pm 11} = -X_{\pm 12} = D/2 \) and \( Y_{\pm 1k} = \pm r/2 \). Note the stretching is orthogonal to the vortex separation. Fig. 7.6(b) shows the dependence of the spin exchange energy on the vortex stretching for \( D = 20 \xi_s \). These results depend only very weakly on total system size as long as the system size is large, since the four component vortices present will behave like a quadrupole with a rapidly decaying energy density for distances greater than \( D \). We find that for small stretching, \( r < 3 \xi_s \), the energy is quadratic in \( r \). For larger stretching the energy becomes linear in \( r \). Fig. 7.6(c) shows the dependence of the spin exchange energy on the vortex separation \( D \) for \( r = 2 \xi_s \). The dependence follows a logarithmic form very closely over the range of \( D \) values considered. This logarithmic dependence on \( D \) can be argued heuristically by assuming that the spin exchange energy density scales as

\[
\mathcal{H}(x, y) \sim \begin{cases} 
\frac{r^2}{x^2+y^2}, & x^2 + y^2 < D^2, \\
0, & x^2 + y^2 > D^2.
\end{cases}
\]

Integrating over this energy density will give a logarithmic dependence on \( D \). This logarithmic dependence on \( D \) is not included in the analytic model (7.35), (7.36).

The spin exchange energy in Fig. 7.6 increases as a function of the vortex stretching \( r \). This means that the \( \psi_{\pm 1} \) vortices of a PCV are confined i.e. they cannot form an unbound state. Fig. 7.5(b) suggests that as the \( \psi_{\pm 1} \) vortices separate, spin waves carry away energy so that the stretching is reduced. However there is still an average increase in stretching as the PCVs
collide. If this stretching continues to increase, the spontaneous creation of a PCV vortex antivortex pair will become favourable so that the spin exchange energy is reduced. This is analogous to color confinement in quantum chromodynamics [421]. To estimate the stretching required to observe this effect, we note that for large stretching Fig. 7.6 predicts that the spin-exchange energy will scale as \((\hbar^2 n_0/M) r/\xi_s\). The energy required to spontaneously create a PCV vortex antivortex pair scales as \((\hbar^2 n_0/M) \ln (D/\xi_s)\). For \(r/\xi_s \gtrsim \ln (D/\xi_s)\), the spontaneous creation of a PCV vortex antivortex pair could lower the system’s free energy. The stretching may reach a maximum distance, however, beyond which any excess energy goes into spin wave production [268]. It is possible that this could occur before the spontaneous creation of a PCV vortex antivortex pair. Reaching this large stretching regime for two PCVs in a setup like in Fig. 7.3 would require a large initial PCV separation so that the vortex stretching can become sufficiently large. Starting the dynamics with a non-zero stretching could allow more modest initial PCV separations to be used, although it may be hard to stabilise such an initial condition against decay through spin wave production.

7.5 Conclusion

In this chapter we have applied a variational Lagrangian approach to derive the model of PCV dynamics introduced by Turner [268]. We compared this model to simulations of a PCV vortex antivortex pair and find semi-quantitative agreement. The distinguishing feature of the PCV dynamics is a vortex mass, which arises from interaction effects within the PCV core. The Turner model can be obtained by approximating this interaction as a quadratic potential which confines the \(\psi_{\pm 1}\) components of a PCV. However, our numerics reveal higher order dynamic effects, which we suspect arise from core interactions coupling the PCV dynamics to spin waves. For high PCV energies, the nature of this coupling to spin waves may have a substantial effect on the confinement of the \(\psi_{\pm 1}\) components. The Lagrangian formulation presented here paves the way for a more detailed study of the PCV core structure and dynamics, which can then be used to extend the Turner model. For example, ansatzes for density profiles of the spin components within the core could be included, as has been done for half-quantum spin vortices [269, 271]. The PCV mass depends on the quadratic Zeeman energy \(q\), both explicitly in Eq. (7.34) and implicitly through the parameter \(a\) from Eq. (7.30). This opens up the possibility of exploring PCV dynamics with a spatially dependent \(q\), which could be done using the Lagrangian formulation presented here. It would also be interesting to explore the effects of a non-zero net \(F_z\) magnetization on...
the PCV dynamics.

We have restricted our study to uniform systems for the purpose of characterising the interaction between PCVs. It is known that a scalar vortex in a harmonic trap can exhibit a precession around the trap centre [115]. It would be interesting to consider the effects of an harmonic trap on the dynamics of a PCV, where not only the density but also the spin properties (through $q_0$) vary spatially.

It should be possible to observe the dynamics of PCVs in current experiments using magnetization sensitive imaging (e.g. see [223, 298]). Indeed, recent experiments in antiferromagnetic spin-1 condensates [272, 273] have been able to prepare half-quantum spin vortices and monitor their subsequent evolution and annihilation. For the ferromagnetic system PCVs can be spontaneously generated in a low-temperature quench from the polar phase to easy-plane phase (see [223, 224, 226, 229] and Fig. 7.1), where the evolution of these vortices then determines the easy-plane phase ordering dynamics (see Chap. 4).
Chapter 8

Conclusion

8.1 Summary of results

In this thesis we have explored a number of aspects of phase ordering in a ferromagnetic spin-1 condensate. The phase ordering follows a quench of the quadratic Zeeman energy, which drives the system from a polar phase to one of three ferromagnetic phases: an easy-axis phase; an easy-plane phase; or an isotropic phase. In Chap. 4 we explored phase ordering in the easy-axis and easy-plane phases. In the easy-axis phase we identified scale invariant coarsening of domains, with a domain size that grows as $L(t) \sim t^{2/3}$. This is consistent with the scale invariant dynamics being driven by inertial hydrodynamics. Separating the domains are sharp domain walls, which appear as a Porod tail in the structure factor of the order parameter. In the easy-plane phase we again identified scale invariant coarsening of domains, with a domain size that grows as $L(t) \sim (t/\log t)$. A logarithmic correction to scaling is familiar from coarsening in other two dimensional systems with a two component order parameter. The scale invariant dynamics is consistent with the model E dynamic universality class, which describes the coarsening of a two component order parameter dynamically coupled to a second conserved field. In our case this second conserved field is the axial magnetization. We find that the coarsening of domains in the easy-plane phase is associated with the annihilation of polar-core spin vortices. The scaling of the coarsening dynamics is consistent with a model of polar-core spin vortex dynamics, as well as a model of spin wave dynamics. In analysing the structure factor of the order parameter, we find a non-integer Porod tail, which is evidence of fractal boundaries of the coarsening domains. We confirm this fractal structure using a box counting algorithm. Finally, we explored thermalisation of the polar Bogoliubov excitation branch during the coarsening dynamics in
the easy-axis phase, and found that the thermalisation adiabatically follows the growth of local order parameter density in the system.

In Chap. 5 we explored phase ordering in the isotropic phase. The ground state manifold of this phase is SO(3) and supports $\mathbb{Z}_2$ vortices. The growth of SO(3) order can be probed by looking at the growth of both spin order and coherence in the $\psi_0$ spinor component (“gauge order”). We find that order in both cases grows as $L(t) \sim t^{1/2}$. The growth of order is consistent with the scaling of a Landau-Lifshitz equation, which can be shown to describe the dynamics of the spin at long wavelengths. We also identify the presence of $\mathbb{Z}_2$ vortices, which initially decay in a way consistent with the growth of order. For later times, residual $\mathbb{Z}_2$ vortices persist in clumps without affecting the growth of spin or gauge order. We also explore coarsening for small nonzero quadratic Zeeman energy, which modifies the ground state to either an easy-axis or easy-plane phase. We find that when the domain energies are large compared to the quadratic Zeeman energy, spin rotations away from the ground state are allowed and the coarsening occurs as in the isotropic phase. For domain energies small compared to the quadratic Zeeman energy, spin rotations away from the ground state are suppressed and the coarsening occurs as it would for nonzero quadratic Zeeman energy. The dynamic transition between these two regimes occurs at a well defined domain size that can be obtained by equating the kinetic energy of the domains to the quadratic Zeeman energy.

In Chap. 6 we explored the very long time phase ordering dynamics in the easy-plane phase. We find that even long after all topological defects (polar-core spin vortices) have annihilated, transverse spin correlations are not thermalised at long wavelengths. We identify that transverse spin wave energy cascades very slowly from long to short wavelengths. At short wavelengths the spin wave energy is able to thermalise. This turbulent cascade moves to longer wavelengths in a scale invariant way as time progresses, and determines the time scale of the very long time phase ordering. The scaling of the energy cascade for early times agrees with an analytic model of weak wave turbulence. We also carried out grand canonical simulations of easy-plane thermalisation, where the condensate is coupled to a reservoir of energy and particles. In this grand canonical case, a persistent energy cascade is not present and the time scale of ordering in the system is set by the annihilation of topological defects.

In Chap. 7 we explored a point vortex model of polar-core spin vortex dynamics. This model predicts that polar-core spin vortices should behave like massive charged particles in two dimensions, and was originally developed by Ari Turner using hydrodynamic arguments. The mass of the vortex arises from the spin exchange energy in the core. We derive this model using a
variational Lagrangian method and test the model against Gross-Pitaevskii simulations. We find qualitative agreement between the analytic model and simulations. The discrepancy between the model and simulations likely arises from a coupling to spin waves and/or dynamic core effects not accounted for in the model. The Lagrangian method we develop could be extended to include a more accurate treatment of the vortex core and spin interaction energy. Including the effect of coupling to spin waves exactly would likely be difficult, however a phenomenological treatment may be possible.

8.2 Ideas for further research in ferromagnetic spin-1 condensates

There are some obvious possible extensions of the work presented in this thesis. In this work we consider only condensates in flat bottomed traps. Many experiments are instead done using harmonic traps, and so it is of interest to consider the effects of an harmonic trap on the phase ordering dynamics, although for the case of a scalar Bose gas it has been argued that a harmonic trap does not change the dynamic critical exponent during phase ordering [352]. We also focus on quasi-two dimensional condensates. Three dimensional or even one dimensional condensates could also be studied, in which case the nature of ordering and behaviour of topological defects is in general expected to be different. In addition to these generalisations, the work in this thesis has inspired a number of other research questions regarding the ordering properties and topological defects in ferromagnetic spin-1 condensates. We describe these below. Some of these ideas we have started exploring.

Phase ordering in the easy-plane phase with nonzero net \( F_z \) magnetization

The results on phase ordering in the easy-plane phase in Chap. 4 considered only systems with no net \( F_z \) magnetization. It would be interesting to consider the role that a nonzero \( F_z \) would have on the phase ordering. With nonzero \( F_z \) the dynamics of polar-core spin vortices (PCVs) develops scalar vortex properties. For example, in the case of two oppositely charged PCVs, as was considered in Chap. 7, a nonzero uniform net \( F_z \) would result in an additional velocity perpendicular to the vortex separation, as one of the \( \psi_{\pm 1} \) components would move more strongly than the other (simulations confirm this conjecture). This would compete with vortex attraction, and eventually
dominate for large net $F_z$. In this limit the phase ordering would be expected to be that of a scalar BEC.\footnote{This behaviour is also relevant for considering the point vortex dynamics of $Z_2$ vortices, which would exhibit both PCV and scalar vortex properties depending on how the phase circulation is distributed between a rotation around the spin axis and a rotation around a direction perpendicular to the spin axis.}

### The effects of a reservoir on phase ordering

The results in this thesis have mainly focused on microcanonical evolution. Indeed this was one of the distinguishing features of our work, as many previous studies of phase ordering have considered only systems coupled to the environment. It would be interesting to consider the effect of a reservoir on the phase ordering dynamics investigated here.

Coupling the system to a reservoir leads to viscous effects in the superfluid flow [175]. In a classical binary fluid, such viscous effects are known to affect phase ordering on intermediate time scales, before a crossover to inertial hydrodynamic growth [37, 38, 55]. It would be interesting to see if the same effect can be observed in the easy-axis phase considered in Chap. 4. Indeed, initial simulations suggest that this is the case, although more careful simulations would be necessary to confirm this.

We have recently carried out simulations of vortex driven phase ordering dynamics in the easy-plane phase that includes coupling to a reservoir. Our tentative results suggest that the phase ordering is slowed by coupling to a reservoir, with domains growing as $L(t) \sim t^{1/2}$. Similar domain growth is observed in the classical $XY$ spin model [37]. The reservoir coupled (grand canonical) dynamics do not conserve total $F_z$ magnetization, which distinguishes the model E dynamics of spin waves considered in Sec. 4.3.4 from model A dynamics of spin waves that occurs in the $XY$-model [49]. Furthermore, with an imbalance in $F_z$ magnetization, one would expect that the PCVs would develop scalar vortex properties, as one of the $\psi_{\pm 1}$ spin components would move more strongly with the imbalance of $F_z$, as discussed above. In addition, damping provides a mechanism for vortex antivortex PCV pairs to lose energy and annihilate, in addition to their stretching, which is similar to the mechanism that vortices in the $XY$-model lose energy [43]. Such changes in vortex dynamics could affect the growth of domains.

In Chap. 5 we presented results for damped phase ordering in the isotropic phase. No apparent differences in the scaling of correlations was observed from these simulations, so it is possible that the isotropic phase ordering is robust against grand canonical effects.
BKT physics

In Chap. 4 we identified that the annihilation of PCVs drives the phase ordering in the easy-plane ferromagnetic phase. For a spin density confined to the transverse plane, the kinetic energy of a single PCV in two dimensions with unit charge scales as the logarithm of the system size $l$:\textsuperscript{2}

$$E \sim \frac{\pi \hbar^2 n_0}{2M} \left(1 - \frac{q}{q_0}\right) \ln \frac{l}{\xi_s}. \tag{8.1}$$

Vortices with a logarithmic dependence on energy such as (8.1) are expected to undergo a Berezinskii-Kosterlitz-Thouless (BKT) transition. This is estimated to occur at the temperature [407],\textsuperscript{3}

$$T_{\text{BKT}} = \frac{\pi \hbar^2 n_0}{4M} \left(1 - \frac{q}{q_0}\right). \tag{8.2}$$

This is not the complete story for PCVs, however, as spin rotations out of the transverse plane allow a PCV to change form. Such spin rotations will be allowed for temperatures above some temperature $T_q$. We expect $T_q$ to increase linearly with $q$, since $q$ is the energy required for a single atom to rotate out of the transverse plane. A spin rotation changes the PCV topology to that of $Z_2$ vortices (the reverse of this change is seen in Fig. 5.4). The existence or nature of a BKT transition of $Z_2$ vortices is still inconclusive [399, 400].

It would be interesting to study BKT transitions in the easy-plane phase as a function of $q$. For small $q_0 - q$ (i.e. small transverse spin density) such that $T_{\text{BKT}} < T_q$, a PCV becomes free before spin rotations out of the plane can occur, and therefore we expect to observe a BKT transition. However, for larger $q_0 - q$ such that $T_{\text{BKT}} > T_q$, the spin density of a PCV is able to rotate out of the plane before a BKT transition can occur, and the PCV has $Z_2$ vortex character. In this regime we expect to see some sort of transition to ensure continuity with the BKT transition at larger $q$. For example, the BKT transition of transverse spin PCVs may evolve into one of $Z_2$ vortices. Alternatively, there may be a topological phase transition between transverse spin PCVs and $Z_2$ vortices. A schematic of the possible $T$–$q$ phase diagram is shown in Fig. 8.1.

\textsuperscript{2}To see this, one computes the kinetic energy of the state (7.3), integrating in polar coordinates from $\xi_s$, which is the characteristic size of the vortex core, to $l$.

\textsuperscript{3}The argument for this is very simple. The multiplicity of a single vortex is the number of points in space to put it, which is $(l/\xi_s)^2$. Therefore the entropy of a single vortex is $S = 2 \ln(l/\xi_s)$. If the energy scales as $\alpha \ln(l/\xi_s)$ then there is a zero point in the free energy $F = E - TS$ at the temperature $T = T_{\text{BKT}} = \alpha/2$. 

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Figure 8.1: Schematic of the expected phase diagram for magnetic order in the easy-plane ferromagnetic phase. For small $q_0 - q$ the PCVs undergo a conventional BKT transition (blue line). For larger $q - q_0$, spin rotations out of the transverse plane can occur before the polar-core spin vortices unbind, and the nature of the vortices changes to $Z_2$ vortices. Spin rotations will be allowed for temperatures above some temperature $T_q$, which we expect will increase linearly with $q$ (red dashed line). The nature of the ordering around $T_q$ and above is unknown.
It would also be interesting to compare a thermal phase transition to the spin ordered phase with the Bose-Einstein condensation transition, which results in superfluid coherence. For example, one could compare the formation of spin order with the formation of $\psi_0$ coherence. Previous theoretical work has shown that spin order can form in a ferromagnetic spin-1 condensate above the condensation transition when the spin interaction energy is comparable to the density interaction [422]. Phase ordering in the $\psi_0$ component could also be explored and compared to the coarsening dynamics of spin order.

Spinodal decomposition in the easy-axis phase

In addition to exploring the stability of magnetic order as a function of temperature in the easy-plane phase, one could also explore the stability of magnetic order in the easy-axis phase as a function of temperature. In this case a temperature induced transition corresponds to a spinodal decomposition line. Again, an additional phase transition may emerge when $q \to 0$ and spin rotations away from the axial direction can occur. We have not yet determined an estimate for the spinodal decomposition line, however such an estimate may be possible using the common technique of Cahn and Hilliard [423]. Like in the easy-plane phase, we could also compare the spinodal decomposition line with the condensate transition. In this case the condensate phase coherence is in the $\psi_{\pm 1}$ components. A comparison of the ordering dynamics of this phase coherence with the ordering of spin in the easy-axis phase was recently done by another student in our group [377].

SU(2) ordering

In addition to the ferromagnetic phases considered in this thesis, a spin-1 condensate can also (theoretically) exist in a phase with SU(2) symmetry by setting the spin interaction to zero. We note that a very recent experimental method has been proposed to control spin interactions in alkali spinor condensates [424]. Alternatively, an SU(2) symmetry can be obtained in a conventional binary condensate by setting the interspecies and intraspecies interactions to be equal [405]. The manifold SU(2) is diffeomorphic to the manifold $S^3$ (the 3-sphere in four dimensions). It would be interesting to compare the ordering properties of this SU(2) phase with the SO(3) phase studied in Chap. 5. The manifold SO(3) is diffeomorphic to $S^3$ with diametrically opposite points identified. The manifold SO(3) supports $\mathbb{Z}_2$ vortices whereas $S^3$ does not, therefore SO(3) is in some sense as close to $S^3$ as one can get while still allowing for $\mathbb{Z}_2$ vortices. Comparing the ordering properties of the SO(3) and SU(2) phases would highlight the role that $\mathbb{Z}_2$ vortices play.
in the phase ordering, either in the dynamics or in the equilibrium state reached. In addition, the ordering properties of $S^n$ systems are of interest in their own right. For example, it is thought that the Heisenberg model in two dimensions, which has an $S^2$ ground state manifold, does not support an ordered phase at any nonzero temperature (e.g. see [425]).

**Ground states and confinement in polar-core spin vortices**

Our study of PCV dynamics in Chap. 7 revealed that the separation of the $\psi_1$ and $\psi_{-1}$ components of a single PCV (vortex “stretching”) plays a crucial role in the dynamics. The lowest energy state of static PCVs has been explored [406]. It would be interesting to extend these studies to include stretched PCVs, since these play an important role in the vortex dynamics. Besides exploring this problem numerically, it may also be possible to study the ground state phase profile of a stretched PCV analytically [426].

The spin interaction energy of a PCV increases with stretching. This means that the $\psi_{\pm1}$ components are confined, i.e. they cannot be separated with finite energy (compare electrically bound charged particles in three dimensions, which can be separated with finite energy). Confining forces are believed to play an important role in the physics of quarks in colour chromodynamics [421]. However, studies of the physics of quarks is difficult, and work has been done to explore confinement in simpler condensed matter systems (e.g., [427]). It would be interesting to explore the role that confinement plays in PCV dynamics, for which both theoretical and experimental studies are tractable.

**Spin structure within easy-axis domain walls**

It would be interesting to study the microscopic structure of the domain walls in the easy-axis phase. In conventional easy-axis magnets, the magnetization can change sign across a domain wall in two ways: either by rotating parallel to the domain wall or by rotating perpendicular to the domain wall. The first is called a Bloch wall; the second a Néel wall. Properties of easy-axis domain walls have been studied in thin ferromagnetic films, both theoretically (e.g., [428–431]) and in experiments (e.g., [432–435]). It would be interesting to compare these results to the structure of easy-axis domain walls in a ferromagnetic spin-1 condensate. Note that the structure of domain walls may be able to be imaged in experiments with spin-1 condensates using high resolution imaging.
Microscopic dynamics of easy-axis vortices

In our studies of easy-axis phase ordering we identified gauge vortices in the $\psi_{\pm 1}$ domains. This inspired a recent study by another student in our group of the phase ordering dynamics of gauge degrees of freedom on top of the spin ordering [377]. A microscopic study of the gauge vortices could also be carried out. From simulations, we observe that the vortex dynamics well within a domain appears to resemble that of ordinary scalar vortices, with oppositely charged vortices moving parallel and same charged vortices rotating about each other. This is as expected, since well within a domain the system resembles a scalar condensate. However, we observe that novel dynamics can appear as vortices approach a domain wall, in which case the full spinor degrees of freedom become important. As a vortex approaches a domain wall, it would be interesting to consider the interplay between the vortex dynamics and the microscopic spin structure of the domain wall. We note that a study of such vortex dynamics may also be amenable to current experiments [272, 273, 436].

8.3 Final remarks

In this thesis we have explored many facets of phase ordering dynamics in a spin-1 ferromagnetic condensate. This system has proved ideal for our studies, which have exposed novel results on scale invariant dynamics, topological defects and thermalisation. Experimental studies of spin-1 condensates have provided elegant and illuminating results on nonequilibrium dynamics; it is hoped that some of the ideas in this thesis could be explored in such experiments.

There are many further aspects of nonequilibrium dynamics in ferromagnetic spin-1 condensates that could be explored. Some of these we have described in Sec. 8.2, however no doubt many others exist. In addition to ferromagnetic spin-1 condensates, there are antiferromagnetic spin-1 condensates. Phase ordering in the antiferromagnetic case has very recently been explored both theoretically [437] and experimentally [281]. Beyond spin-1 there are higher spin condensates, which support highly nonclassical symmetry states and topological defects. Such systems will provide an almost endless arena to explore phase ordering dynamics, topological defects, and both equilibrium and nonequilibrium physics in general. We hope that the results in this thesis will help motivate and facilitate such further studies.
Appendix A

Free energy of a closed system

In Sec. 2.1 we obtained expressions for the free energy and partition function by considering a system coupled to a reservoir. In the case of a large system, we can adopt an alternative perspective that does not require the use of a reservoir. In this perspective, the partition function is defined as the Laplace transform of the multiplicity function \( \Omega = \exp(-S) \) [305],

\[
Z = \int dV_1...dV_n \Omega(V_1, ..., V_n) \exp \left( - \sum_i \beta_i V_i \right).
\]

(A.1)

The partition function \( Z \) in this form can be understood as a generating function of the moments of \( \Omega \) [438] and is often easier to compute than the multiplicity function [305]. The multiplicity function counts the number of microstates corresponding to each macrostate with fixed \( V_1, ..., V_n \), and so Eq. (A.1) is equivalent to Eq. (2.3). The free energy is then defined as \( F = -\ln Z \) (the generating function of the cumulants of \( \Omega \)). We can now write the multiplicity function as an inverse Laplace transform,

\[
\exp(S) = \Omega = \int_C d\beta_1...d\beta_n \exp \left( -F(\beta_1, ..., \beta_n) + \sum_i \beta_i V_i \right),
\]

(A.2)

where \( C \) is the appropriate contour in the complex plane. The quantity \( -F(\beta_1, ..., \beta_n) + \sum_i \beta_i V_i \) in Eq. (A.2) is extensive, and therefore for a large system we can approximate the integral using the method of steepest descent. In the limit of infinite system size, the integral can be approximated by the value of the integrand at the saddle point [439]. This gives,

\[
F(\beta_1, ..., \beta_n) = -S(V_1, ..., V_n) + \sum_i \beta_i V_i.
\]

(A.3)

This relationship relates the multiplicity (through \( S \)) to the generally simpler partition function (through \( F \)). The unique choice of the \( \beta_i \) correspond to
the saddle point in evaluating Eq. (A.2), which occurs when
\[ \frac{\partial F}{\partial \beta_i} = V_i. \]  
(A.4)
Furthermore, using Eqs. (2.1) and (A.4) we obtain that,
\[ \frac{\partial}{\partial V_j} \left( -S(V_1, ..., V_n) + \sum_i \beta_i (V_1, ..., V_n) V_i \right) = \sum_i V_i \frac{\partial \beta_i}{\partial V_j} = \frac{\partial F}{\partial V_j}. \]  
(A.5)
Therefore the minima of \(-S + \sum_i \beta_i V_i\) coincide with those of \(F\) (with respect to the \(V_i\)) so that the result (A.3) is equivalent to Eq. (2.2). Note the difference, however, between this approach and the reservoir approach. In the reservoir approach, the \(\beta_i\) are fixed and the value of the free energy is a stationary point under variation of the \(V_i\). This follows from considering the entropy of both the system and the reservoir. In the Laplace transform approach, we compute \(F\) from the partition function, and relate this to the multiplicity with a fixed \(V_i\) using the method of steepest descent, which picks out the optimal \(\beta_i\). Here no consideration of a reservoir is necessary.

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Appendix B

Proof that the functional solution of $f(x_1 x_2) = f(x_1) f(x_2)$ is a power law

Consider a function $f : \mathbb{R}^+ \to \mathbb{R}^+$, where $\mathbb{R}^+$ is the set of positive real numbers, with the property

$$f(x_1 x_2) = f(x_1) f(x_2).$$  \hfill (B.1)

We also suppose that $f$ is invertible.\footnote{In the context of an RG transformation, $f(x)$ would not represent the transformation on the parameters in the Hamiltonian (such a transformation may not be invertible). Instead, $f(x)$ would represent the dependence of the transformation on the rescaling factor.} We can introduce a second function $g(x) \equiv \ln f(x)$. This function satisfies

$$g(x_1 x_2) = g(x_1) + g(x_2)$$  \hfill (B.2)

and is also invertible, since both the logarithm function and $f$ are invertible. We define the inverse of $g$ as $G$. The function $G$ satisfies

$$G(y_1 + y_2) = G(y_1) G(y_2).$$  \hfill (B.3)

We therefore have that

$$G(y + h) - G(y) = G(y) (G(h) - G(0)).$$  \hfill (B.4)

Dividing by $h$ and taking the limit $h \to 0$ gives

$$G'(y) = G(y) G'(0).$$  \hfill (B.5)
Therefore $G(y) = Ae^{G'(0)y}$ for some constant $A$. This means that $g$ takes the form

$$\ln f(x) = g(x) = \ln(x/A)/G'(0).$$

(B.6)

We then obtain that $f(x) = Bx^\alpha$ where $B = 1/A^{1/G'(0)}$ and $\alpha = 1/G'(0)$ QED.
Appendix C

A model of weak wave turbulence in the easy-plane ferromagnetic phase

In this appendix we apply the theory of weak wave turbulence to the case of an easy-plane ferromagnetic condensate. From this analysis we obtain energy cascades $k^{-1/3}$ and $k^{-4/3}$, consistent with that observed for early times in the compressible spectrum in Fig. 6.3. The derivation (and in general any weak wave turbulence derivation) proceeds by deriving a Hamiltonian in terms of the Fourier coefficients of the relevant excitations. In the case of weak interactions, we can separate the Hamiltonian into a part that drives fast free evolution and a part that drives a slow interaction. The scaling of the turbulent cascade can be shown to be determined by the system dimension $D$, the spectrum of the free evolution, the number of waves that participate in the dominant interaction and the wavenumber scaling of the coefficient, or vertex, of this interaction. The general result is (see [408, 440] for a derivation$^1$),

$$\epsilon(k) \sim \begin{cases} k^{1+\nu-\mu-D}, & \text{3-wave interaction,} \\ k^{1+\nu-2\mu/3-D}, & \text{4-wave interaction.} \end{cases}$$

where $\nu$ is the wavenumber scaling of the free particle spectrum and $\mu$ is the scaling of the interaction vertex. We wish to find the scaling parameters $\nu$

$^1$The analytic derivation presented in these references involves finding a fixed point of the equation of motion for the expectation value of the excitation spectrum, which takes the form of a power law cascade. This requires making what is known as the random phase approximation, which is valid for fast free evolution and gives rise to correlators of excitations obeying Gaussian statistics. Higher order correlations can then be decomposed into pair correlators using Wick’s theorem. This approximation gives a closed set of equations of motion, i.e. avoids a set of equations that couples to higher and higher order correlators. Other closure techniques also exist in turbulence, see for example [441].
and $\mu$ for weak wave turbulence in the easy-plane phase.

We begin with the state (6.2) and allow fluctuations in $\theta$ and $\phi$ around the ground state, since these fluctuations are the most important in the low energy limit (see the discussion above Eq. (6.5)). We will find that interactions of these fields give rise to the $k^{-1/3}$ cascade for $k > 1/\xi_s$ and a $k^{-4/3}$ cascade for $k < 1/\xi_s$. In terms of the two fields $\theta$ and $\phi$, we can write the Lagrangian density for the system as ($\hbar \equiv 1$),

$$\mathcal{L} \equiv \sum_{m} \left( \text{Im} \left( \psi_{m}^{*} \frac{\partial \psi_{m}}{\partial t} \right) + \psi_{m} \frac{1}{2M} \nabla^{2} \psi_{m} - q \frac{m^{2}}{2} |\psi_{m}|^{2} \right) - \frac{g_{n}}{2} n^{2} - \frac{g_{s}}{2} |\mathbf{F}|^{2}$$

$$= \frac{\partial \theta}{\partial t} n_{0} \sin^{2} \beta \cos 2\phi - \frac{n_{0} \sin^{2} \beta}{2M} \left( |\nabla \theta|^{2} + |\nabla \phi|^{2} \right) - \frac{g_{s} n_{0}^{2}}{2} \sin^{4} \beta \cos^{2} 2\phi$$

$$- g_{s} n_{0}^{2} \sin^{2} \beta \cos^{2} \beta \sin 2\phi,$$  \hspace{1cm} (C.2)

where we have dropped constant terms in the second line. Evidently the transverse spin angle $\theta$ is conjugate to $p \equiv F_{z} = n_{0} \sin^{2} \beta \cos 2\phi$, as expected since $F_{z}$ is the generator of transverse spin rotations. The important next step is to rewrite the Lagrangian density entirely in terms of the conjugate variables $\theta$ and $F_{z}$. This requires the identity,

$$|\nabla \cos 2\phi|^{2} = 4 \sin^{2} 2\phi |\nabla \phi|^{2} = 4 \left(1 - \cos^{2} 2\phi\right) |\nabla \phi|^{2}.$$  \hspace{1cm} (C.3)

Therefore,

$$|\nabla \phi|^{2} = \frac{|\nabla p|^{2}}{4 \sin^{2} \beta - 4p^{2}}.$$  \hspace{1cm} (C.4)

We then obtain that,

$$\mathcal{L} = \frac{\partial \theta}{\partial t} p - \frac{n_{0} \sin^{2} \beta}{2M} |\nabla \theta|^{2} - \frac{|\nabla p|^{2}}{8M n_{0} \sin^{2} \beta} \left(1 - \frac{p^{2}}{n_{0}^{2} \sin^{4} \beta} \right)^{-1} - \frac{g_{s}}{2} p^{2}$$

$$- g_{s} n_{0}^{2} \sin^{2} \beta \cos^{2} \beta \sqrt{1 - \frac{p^{2}}{n_{0}^{2} \sin^{4} \beta}}.$$  \hspace{1cm} (C.5)

The square root in the spin interaction follows from replacing $\sin 2\phi$ with $\cos 2\phi$. The positive root can be taken since $\sin 2\phi$ will be close to one for small fluctuations of $\phi$, i.e. small $F_{z}/F_{\perp}$.

The system can now be written in Hamiltonian form, with equations of motion,

$$\frac{\partial \theta(x)}{\partial t} = - \frac{\delta}{\delta p(x)} \int d^{2}x' \mathcal{H}(x'),$$

$$\frac{\partial p(x)}{\partial t} = \frac{\delta}{\delta \theta(x)} \int d^{2}x' \mathcal{H}(x'),$$  \hspace{1cm} (C.6)
and a Hamiltonian density,

\[ \mathcal{H} = n_0 \sin^2 \beta \frac{2}{M} |\nabla \theta|^2 + \frac{p^2}{8Mn_0 \sin^2 \beta} \left( 1 + \frac{p^2}{n_0^2 \sin^4 \beta} + \ldots \right) + \frac{g_s}{2} p^2 - g_s n_0^2 \sin^2 \beta \cos^2 \beta \left( \frac{p^2}{2n_0^2 \sin^4 \beta} - \frac{p^4}{4n_0^4 \sin^8 \beta} + \ldots \right) \]

\approx n_0 \sin^2 \beta \frac{2}{M} \left( |\nabla \theta|^2 + |\nabla \rho|^2 \right) - 2g_s n_0^2 \sin^2 \beta \cos 2\beta \rho^2 + \frac{2n_0 \sin^2 \beta}{M} |\nabla \rho|^2 \rho^2

+ 4g_s n_0^2 \sin^2 \beta \cos^2 \beta \rho^4. \tag{C.7} \]

In the second equality we have expanded to quadratic order in \( p \), since \( \cos 2\phi \) is small for small fluctuations of \( \phi \). We have also introduced the scaled momentum \( \rho \equiv p/(2n_0 \sin^2 \beta) \) for convenience. The Hamiltonian density (C.7) consists of three terms: a free evolution term,

\[ \mathcal{H}_f = n_0 \sin^2 \beta \frac{2}{M} \left( |\nabla \theta|^2 + |\nabla \rho|^2 \right) - 2g_s n_0^2 \sin^2 \beta \cos 2\beta \rho^2, \tag{C.8} \]

and two interaction terms,

\[ \mathcal{H}_1 = \frac{2n_0 \sin^2 \beta}{M} |\nabla \rho|^2 \rho^2 \tag{C.9} \]

arising from the nonlinear transformation to the conjugate variables \( \theta \) and \( F_z \), and

\[ \mathcal{H}_2 = 4g_s n_0^2 \sin^2 \beta \cos^2 \beta \rho^4 \tag{C.10} \]

arising from the spin interactions in the condensate.

To write Eq. (C.7) in terms of wavemixing processes, we introduce the complex field,

\[ A \equiv \sqrt{n_0} \sin \beta (\theta + i\rho). \tag{C.11} \]

We write this as a Fourier series,

\[ A = \frac{1}{l} \sum_k b_k e^{ik \cdot x}, \tag{C.12} \]

where \( l \) is the system size.
We firstly consider the noninteracting contribution (C.8). In terms of the Fourier modes \( b_k \), the noninteracting Hamiltonian becomes,

\[
H_f \equiv \int d^2 x \left( \frac{n_0 \sin^2 \beta}{2M} (|\nabla \theta|^2 + |\nabla \rho|^2) - 2g_s n_0^2 \sin^2 \beta \cos 2\rho \right)
\]

\[
= \frac{1}{2M} \sum_k k^2 b_k^* b_k + \frac{q}{4} \sum_k \left( 2b_k^* b_{-k} - b_k b_{-k} - b_k^* b_{-k}^* \right)
\]

\[
= \sum_{k>0} \left( \frac{k^2}{2M} + \frac{q}{2} \right) \left( b_k^* b_k + b_{-k}^* b_{-k} \right) - \frac{q}{2} \sum_{k>0} \left( b_k b_{-k} + b_k^* b_{-k}^* \right), \quad (C.13)
\]

where the sum \( \sum_{k>0} \) denotes a sum over one half of \( k \)-space. The noninteracting Hamiltonian can now be diagonalised via a Bogoliubov transformation [356] to obtain,

\[
H_f = \sum_k \omega_k a_k^* a_k, \quad (C.14)
\]

where

\[
\omega_k = \sqrt{\frac{k^2}{2M} \left( \frac{k^2}{2M} + q \right)} \quad (C.15)
\]

is the free particle dispersion relation and gives the time scale of free particle evolution. The quasiparticle amplitudes \( a_k \) are defined by,

\[
b_k = u_k a_k - v_k a_{-k}^*, \quad (C.16)
\]

with

\[
u_k^2 = \frac{1}{2} \left( \frac{\epsilon_k + q}{\omega_k} + 1 \right), \quad v_k^2 = u_k^2 - 1. \quad (C.17)
\]

We now want to rewrite the interaction terms in terms of the new variables \( a_k \) and \( a_k^* \). For the first interaction (C.10) we have,

\[
\frac{2n_0 \sin^2 \beta}{M} \int d^2 x \left| \nabla \rho \right|^2 \rho^2 = \frac{1}{8M n_0 t^4 \sin^2 \beta} \int d^2 x \left| \sum_k k b_k e^{ik \cdot x} + \sum_k k b_k^* e^{-ik \cdot x} \right|^2
\]

\[
\times \left( \sum_k b_k e^{ik \cdot x} - \sum_k b_k^* e^{-ik \cdot x} \right)^2, \quad (C.18)
\]

with

\[
\sum_k b_k e^{ik \cdot x} - \sum_k b_k^* e^{-ik \cdot x} = \sum_k \left( u_k a_k - v_k a_{-k}^* \right) e^{ik \cdot x} - \sum_k \left( u_k a_k^* + v_k a_{-k} \right) e^{-ik \cdot x}
\]

\[
= \sum_k (u_k + v_k) \left( a_k - a_{-k}^* \right) e^{ik \cdot x}. \quad (C.19)
\]
and
\[ \sum_k k b_k e^{ik \cdot x} + \sum_k k b_k^* e^{-ik \cdot x} = \sum_k k (u_k - v_k) (a_k + a_k^*) e^{ik \cdot x}. \] (C.20)

The interaction (C.18) has an interaction vertex
\[ T_{k_1 k_2 k_3 k_4} \sim \frac{1}{M n_0 l^2} k_1 \cdot k_2 (u_{k_1} - v_{k_1}) (u_{k_2} - v_{k_2}) (u_{k_3} + v_{k_3}) (u_{k_4} + v_{k_4}) \]
\[ = \frac{1}{M n_0 l^2} k_1 \cdot k_2 (u_{k_1}^2 - v_{k_1}^2) (u_{k_2}^2 - v_{k_2}^2) (u_{k_3} + v_{k_3}) (u_{k_4} + v_{k_4}) \]
\[ = \frac{1}{M n_0 l^2} k_1 \cdot k_2 (u_{k_3} + v_{k_3}) (u_{k_4} + v_{k_4}) \]
\[ (u_{k_1} + v_{k_1}) (u_{k_2} + v_{k_2}) \]
\[ \sim \frac{1}{M (n_0 l^2)}. \] (C.21)

Conservation of energy and momentum restricts the possible forms of interactions to either
\[ \omega_1 + \omega_3 = \omega_2 + \omega_4, \quad k_1 + k_3 = k_2 + k_4 \] (C.22)
or
\[ \omega_1 + \omega_2 = \omega_3 + \omega_4, \quad k_1 + k_2 = k_3 + k_4. \] (C.23)

For the second interaction (C.10) we have,
\[ 4 g_s n_0^2 \sin^2 \beta \cos^2 \beta \int d^2x \ u^4 = \frac{g_s \cos^2 \beta}{4l^4} \int d^2x \ \left( \sum_k b_k e^{ik \cdot x} - \sum_k b_k^* e^{-ik \cdot x} \right)^4. \] (C.24)

The vertex for this second interaction scales as,
\[ V_{k_1 k_2 k_3 k_4} \sim \frac{g_s}{4l^2} (u_{k_1} + v_{k_1}) (u_{k_2} + v_{k_2}) (u_{k_3} + v_{k_3}) (u_{k_4} + v_{k_4}) \]
\[ \sim \frac{g_s n_0}{4 (n_0 l^2)} \left( \epsilon_k + q \right)^2. \] (C.25)

with the conservation laws (C.22) or (C.23). Both interactions (C.25) and (C.21) are four wave interactions.

The interaction terms are small compared to the free evolution for high atom numbers \( n_0 l^2 \). By comparing the two interaction vertices (C.21) and (C.25), we see that the first interaction dominates the second in the regime \( k \xi_s \gg 1 \), and vice versa for the regime \( k \xi_s \ll 1 \).
We now have all the necessary components to obtain the scaling solution (C.1). For \( k\xi_s \gg 1 \), we use the interaction vertex (C.21), which scales with wavenumber as \( k^2 \) and therefore gives \( \mu = 2 \). In the regime \( k\xi_s \gg 1 \) the spectrum (C.15) is quadratic, \( \omega_k \sim k^2 \), giving \( \nu = 2 \). Therefore Eq. (C.1) gives a cascade,

\[
\epsilon(k) \sim k^{-1/3},
\]

consistent with the early time scaling of the compressible spectrum in Fig. 6.3. It can also be shown that for this choice of \( \mu \) and \( \nu \), the cascade is direct [408]. The steepening of the cascade to \( \epsilon_k \sim k^{-4/3} \) as \( k \) decreases to \( k\xi_s < 1 \) can then be attributed to a flattening of the spectrum (C.15) to \( \omega_k \sim k \). Unfortunately, the regime \( k\xi_s < 1 \) also predicts that the second interaction vertex (C.25) should also be important, which from Eq. (C.1) would appear to give the somewhat unusual scaling \( \epsilon(k) \sim k^{8/3} \), which is defiantly not what we observe in Fig. 6.3. Furthermore, Fig. 6.3 suggests that the cascade steepens beyond \( \epsilon_k \sim k^{-4/3} \) for later times. We have currently not resolved these discrepancies.

Our discussion above also predicts that fluctuations of \( F_z \) should show the same turbulent properties as fluctuations of transverse spin. Fluctuations of \( F_z \) (measured by \( \rho \)) contribute to the noninteracting Hamiltonian (see Eq. C.8) a term (restoring \( \hbar \))

\[
H_\rho = \int d^2x \left( \frac{\hbar^2 n_0 \sin^2 \beta}{2M} |\nabla \rho|^2 + q_0 n_0 \sin^2 \beta \cos 2\beta \rho^2 \right) = \sum_k 2\pi k \epsilon'_\rho(k),
\]

where

\[
\epsilon'_\rho(k) \equiv \left( \frac{\hbar^2 n_0 \sin^2 \beta}{2M} k^2 + q_0 n_0 \sin^2 \beta \cos 2\beta \right) \int \frac{d\Omega_k}{2\pi} |\hat{\rho}|^2,
\]

with

\[
\hat{\rho} = \frac{1}{l} \int d^2x \rho e^{-ik \cdot x}
\]

the Fourier transform of \( \rho \). We plot the scaled spectrum \( \epsilon_\rho(k) \equiv 2\pi k \epsilon'_\rho(k) \) in Fig. C.1 for the same times as displayed in Fig. 6.3(b). The agreement between the scaling in Fig. C.1 and Fig. 6.3(b) is excellent.
Figure C.1: Energy spectrum of $F_z$ fluctuations, Eq. (C.28), showing excellent agreement with the scaling in Fig. 6.3(b).
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